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# Level curvature distribution in the Quantum Hall effect

The parametric dynamics of the energy spectrum in the regime of the integer Quantum Hall effect is studied. The second derivative of the electron levels is calculated with respect to the external magnetic flux. It is shown that the distribution of the level curvatures in the center of the Landau band exhibits universal size-independent behavior. The non-interactive electron gas in two-dimensional disordered systems subject to a strong perpendicular magnetic field is modeled with the goal to explore the response of the energy spectrum to an external perturbation. Physically, the role of the external perturbation can be played by an additional Aharonov-Bohm flux applied parallel to the system. One of the central questions to answer is how the parametric spectral correlation functions are distinct from those belonging to the conventional classes of global universality. It is expected that the level curvature distribution will compose a new unitary class which is specific for the Quantum Hall Effect regime.
***Key words:*** electron conductivity, critical phenomena, two-dimensional electron gas, quantum Hall effect, critical index, energy level statistics, unitary ensemble.

И.Х. Жәрекешев

**Холлдың кванттық эффектісінде энергия деңгейлері қисықтығының үлестіруі**

Холлдың кванттық толықөлшемді эффектісі режиміндегі энергия спектрінің параметрлік қозғалысы зерттелген. Электрондық деңгейлердің сыртқы магниттік ағынының өзгеруіне байланысты екінші туынды орындары есептеп шығарылған. Деңгейлер қисықтығының үлестірілуі Ландау зонасының орталығында әмбебап өлшемді-тәуелсіз күй танытатындығы анықталған. Күшті перпендикулярлы магнит өрісіне ұшыраған екіөлшемді реттелмеген жүйедегі өзара әсер етпейтін электрондық газ модельденген. Энергия спектрінің сыртқы қозуға әсері зерттелген. Физикалық тұрғыда сыртқы ұйыткудың рөлін жүйеге параллель тіркелген Ааронов-Бомның қосымша магниттік ағыны атқаруы ықтимал. Критикалық үлестіруді күнделікті жаһандық универсал кластарға тиісті үлестірулерден айыра тану өзекті мәселелердің бірі болып табылады. Деңгейлер қисықтығының үлестірілуі тек Холлдың кванттық режиміне тән жаңа унитарлық класс болып келетіні күтілуде.
***Түйін сөздер:*** электрондық өткізгіштік, критикалық құбылыс, екіөлшемді электрондық газ, Холлдың кванттық эффектісі, электрондық локализация.

И.Х. Жарекешев

**Распределение кривизны уровней энергии в квантовом эффекте Холла**

Изучается параметрическое движение спектра энергии в режиме целочисленного эффекта Холла. Вычисляется вторая производная положений электронных уровней по отношению к изменению внешнего магнитного потока. Показано, что распределение кривизны уровней в центре зоны Ландау проявляет универсальное размерно-независимое поведение. Электронный невзаимодействующий газ в двумерной неупорядоченной системе, подверженный сильному перпендикулярному магнитному полю, моделируется с целью исследовать отклик спектра энергии на внешнее возбуждение. Физически роль внешнего возмущения может играть дополнительный магнитный поток Ааронова-Бома, приложенный параллельно системе. Один из центральных вопросов является как критическое распределение, отличающее от распределений, принадлежащих к обычным классам глобальной универсальности. Ожидается, что распределение кривизны уровней составляет новый унитарный класс, который специфичен для режима квантового эффекта Холла.
***Ключевые слова:*** электронная проводимость, критические явления, двумерный электронный газ, квантовый эффект Холла, электронная локализация.

**Introduction**

The eigenvalue dynamics of the spectra of disordered systems in dependence of an external parameter has attracted considerable interest in recent years. It has been motivated by the fact that the conductance though the system is closely related with the sensitivity of its eigenvalue spectrum to a perturbation. It this respect the works of Altshuler et. al. [1] should be mentioned in particular, who showed that parametric dynamics of disordered solids is governed by the global universality laws.

Investigating statistics of the parametrical energy-level motion proved to be one of the fundamental issues in mesoscopic physics. The distributions of level velocities and level curvatures exhibit generic behavior which is inherent not only for disordered metals, but also for chaotic dynamical problems, as well as for nuclear and atomic physics. It became clear that study of parametric variations of energetic quantities leads to new, very interesting and unexpected results. Mush of important work has been performed in the past [2-7].

It was shown first by Simons and Altshuler [3-6], that the parametric spectral correlations are independent of properties of the particular system, and of the nature of perturbation. By using the supersymmetry technique for disordered metals they have found a universal form for the expression of the velocity-velocity correlation function:

$$C(x,x',\partial_x\varepsilon_i(x),\partial_x\varepsilon_k(x')) = \langle\partial_x\varepsilon_i(x)\partial_x\varepsilon_k(x')\rangle. \qquad (1)$$

The averaging denoted by б...с may be performed over the interval of the energy spectrum, and/or over a finite range of the external parameter, and/or over the ensemble of random realizations.

This universality holds not only for disordered conductors, but also for the whole variety of non-integrable systems such as a hydrogen atom in a strong magnetic field, chaotic billiards, mesoscopic rings in Aharonov-Bohm flux. Basically, it plays a role of an indicator of quantum chaos. The invariant functional form of the velocity-velocity correlator implies that any statistical property of a set of random functions $e_i(x)$ is entirely determined by type of the Dyson ensemble characterizing the system.

We consider a quantum disordered system which is described by the Hamiltonian

$$\hat{H} = \hat{H}_0 + X\hat{h} \qquad (2)$$

and is subject upon an external perturbation controlled by some parameter X. Physically this parameter can represent a magnetic or an electric field, gate voltage, the strength of the random potential or the shape of the sample edges. The energy spectrum of the system as a function of parameter X disperse into a sequence of bands described by a set of randomly fluctuating functions $E_i(X)$. Generally, this sequence can be identified as a motion of fictitious one-dimensional particles, where the external parameter X plays a role of fictitious time. The parametrical motion of energy levels can be considered as a dynamical problem of these fictitious particles, which interact with each other.

After rescaling the energies with respect to the mean level spacing Δ and modifying the external perturbation parameter X by the generalized conductance C(0) (see the definition below)

$$\varepsilon_i(x) = \frac{E_i(X)}{\Delta}, \qquad x = X\sqrt{C(0)} \qquad (3)$$

the specific information about the sample can be eliminated from the correlation function of the electron density of states. The question arises whether or not the statistical behavior of parameter-induced variations in $e_i(x)$ is sensitive to detailed properties of the system under consideration.

There is a number of important entities characterizing the parametrical dependence of the energy spectrum. One of them is called the level velocity defined as the first derivative of the level position $\varepsilon_i(x)$ with respect to the dimensionless parameter x.

$$\partial_x \varepsilon_i(x) = \frac{1}{\Delta}\frac{\partial \varepsilon_i(x)}{\partial x}. \tag{4}$$

Another one is known as a level curvature determined as the second derivative of $\varepsilon_i(x)$ with respect to x.

$$k_i(x) = \frac{1}{\Delta}\frac{\partial^2 \varepsilon_i(x)}{\partial x^2}. \tag{5}$$

where Δ is the mean level spacing.

Boundary conditions imposed on the electron wave function

$$\Psi(x+L) = \Psi(x)\exp(2\pi i\varphi)$$

can be interpreted as the application of an Aharonov-Bohm flux (AB-flux) ϕ on the system: $\varphi = \phi/\phi_0$, where $\phi_0 = hc/e$ is the flux quantum. Here L is the linear size of system under consideration.

The response of the wavefunction to a change in the phase of the boundary conditions can be quantitatively measured by the zero-flux level curvature determined as a second derivative of the function $E_i(\phi)$ with respect to the AB-flux ϕ:

$$k_i = \frac{1}{\Delta}\frac{\partial^2 E_i(\phi)}{\partial \phi^2}\bigg|_{\phi=0} \tag{6}$$

for an individual energy level $E_i$. Due to an argument by Thouless the properly averaged curvatures can be assumed to be proportional to the conductance of the sample. In what follows we omit the indexing i and consider the curvature as a statistical variable {k}.

**Ergodic regime**

The power-law asymptotic behavior of the probability distribution of the level curvatures for GOE, GUE, and GSE ensembles of the RMT has been predicted by Gaspard et al [8]:

$$P_\beta(k) \propto \frac{1}{|k|^{\beta+2}}, \tag{7}$$

where the `repulsion parameter' b = 1,2 and 4 for the GOE, GUE, and GSE ensembles, respectively.

Concerning the complete functional form, it was conjectured by Zakrzewski and DeLande [9] and proved rigorously by von Oppen [10-11] that the level curvature distribution for the GOE case (b=1) is given by the generalized Cauchy distribution

$$P_{GOE}(k) = \frac{1}{2}(1+k^2)^{-3/2}, \tag{8}$$

which is normalized and symmetric with respect to zero curvature k = 0. Interestingly that all even momemta are divergent $<k^{2n}>\to\infty$, where n > 1 is integer. Since all odd momenta of the distribution Eq. (8) are zeros, it is convenient to deal with the double absolute value *K*: = 2 |k| of the curvature:

$$P_{GOE}(K) = \frac{1}{2}(1+K^2/4)^{-3/2}, |k|, \tag{9}$$

because the typical $K_0$ : = exp<lnK> = 1, while the average <K> = 2. The useful relation between the typical and the average modulo is

$$k_0 := \exp\langle \ln|k| \rangle = \langle |k| \rangle/2. \tag{10}$$

The fractional momenta are given by

$$< K^\alpha > = -\frac{2^\alpha}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{-\alpha}{2}\right), \qquad -1 < \alpha < 2, \tag{11}$$

The variance of the curvature logarithm is constant $\sigma^2(\ln K) = \pi^2/6$.

We start with the Anderson Hamiltonian

$$H = \sum_{i\sigma} \varepsilon_n a^+_{n\sigma} a_{n\sigma} + \sum_{<i,j>,\sigma} V_{i,j} (a^+_{i\sigma} a_{j\sigma} + a_{i\sigma} a^+_{j\sigma}), \tag{12}$$

where the random site energies $\varepsilon_i$ are distributed uniformly in the interval [-W/2,W/2]. $V_{ij}$ is the hopping element between neighboring sites i and j in the lattice. Hopping elements allows one to extract the phase factor, so that Eq. (12) corresponds to the one parameter group of Hamiltonians, which can be expressed in a simple matrix form

$$\hat{H} = \hat{H}_0 + e^{i\varphi}\hat{V} + e^{-i\varphi}\hat{V}^T. \tag{13}$$

We diagonalize numerically the Hamiltonian for a disordered sample with simple cubic structure of various sizes ranging between L = 50 and 200 measured in units of the lattice constant a. The strength of the disordered potential spreads from W = 2 to 10. For each combination {L,W} the number of ensemble realizations is chosen such that at least 20000 eigenvalues have been calculated. The exact diagonalization is performed by the Lanczos algorithm implemented for eigenvalue problem of hermitian matrices.

**Level curvature distribution**

The distribution of level curvatures in the delocalized regime is known to follow the RMT result: $P(k) = (1+k^2)^{-3/2}$ (the orthogonal case), where $k = K\Delta/(\pi\sigma)$ [9]. In the insulating regime, the distribution is logarithmically-normal due to exponential localization of the wave functions [12-14]. Both of the limiting distribution functions have been numerically confirmed for the Anderson model [15,16]. We compute the disorder-induced transition of P(k) which accompanies the AT. We plot the distribution of lnK in Fig.1 for various sizes and three different values of the disorder. Indeed, in the diffusive regime the data for P(lnK) (right curves) are equivalent to each other, except of the shift in <lnK >. After rescaling k = K/exp<lnK>, they coincide with the RMT expression independently of L and W. In the insulating regime, P(lnK) are well approximated by the Gaussian statistics. When approaching the transition, the distribution P(lnK) exhibits critical behavior. For finite sizes it experiences a continuous crossover between the above limits, which is governed by the single scaling variable L/x. At $W=W_c$ all curves fall onto one common L-invariant function, which is found to be well interpolated by

$$P_c(k) = A(1+k^{2/\omega})^{-3\omega/2},$$

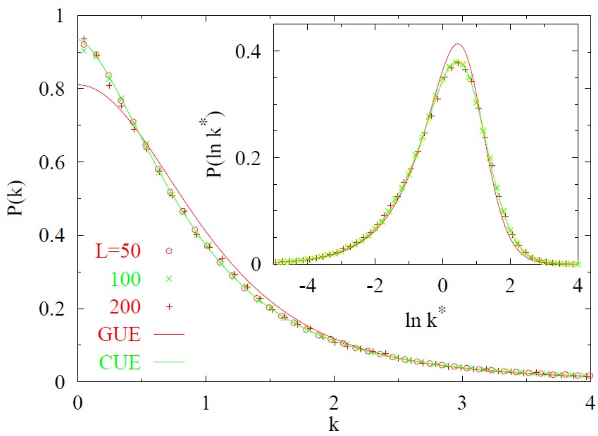


**Figure 1** – Distribution of level curvature P(k) in the lowest Landau level in the quantum Hall effect for various system sizes L (shown by different colours). All points lie on a single curve corresponding to the universality class called Critical Unitary Ensemble (CUE). *Inset*: the same dependence in log-log scale.

with the normalization constant A. The only fitting parameter here proved to equal $\omega \approx 1.2$. A similar formula for the critical $P_c(k)$ has been obtained also in [15], but with a slightly different $\omega$. Note that asymptotic form of $P_c(k)$ for large curvatures is the same as in the metallic regime $P(k) \sim k^{-3}$.

**Scaling of level curvatures.** We focus now on the orthogonal situation, i.e. at zero flux, when the time-reversal symmetry is preserved. First, we calculate the dependence of the mean value of the logarithm of the level curvature $K \in \Delta^{-1}|d^2 E_n/df|_{f=0}$ on the system size and disorder. The data have been averaged over many realizations and over central part of the spectrum, so that about $10^4$ curvatures are obtained for each pair of L and W. The results demonstrate behavior of the geometric mean of the curvature which is typical for the conductance in accordance with the Thouless conjecture [17]. In the insulating regime $<\ln K> \sim -L/x$, where x is the localization length, while on the metallic side Ohm's law $\exp<\ln K> \sim L/W^2$ is obtained. At $W = W_c$ the value $<\ln K>$ is not sensitive to L. In other words, a common intersection of curves for different sizes indicates the position of the mobility edge. Our results are similar to those in papers [15,16].

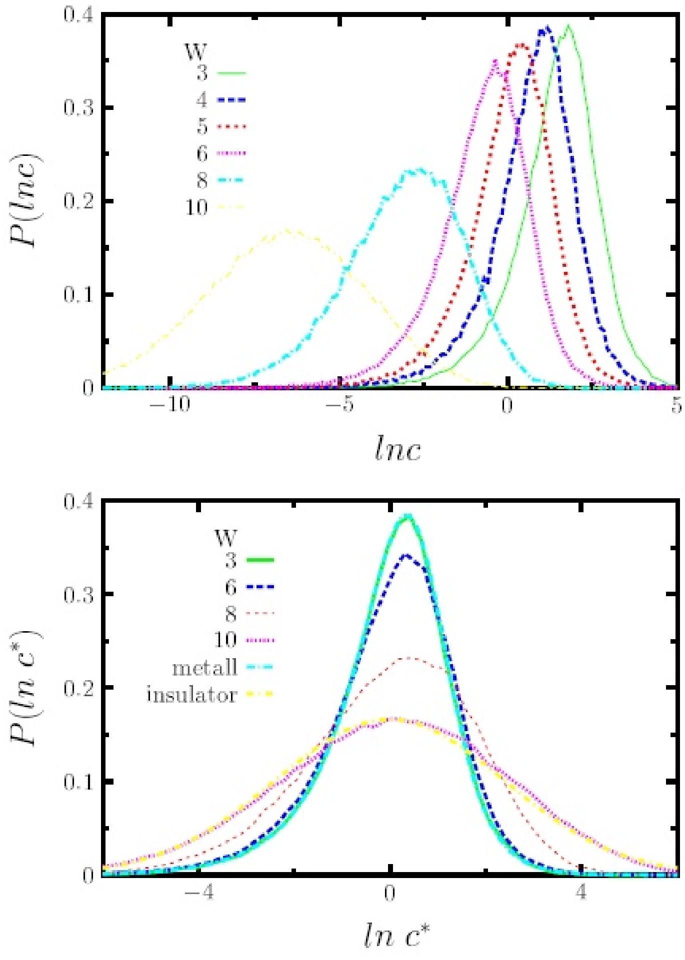


**Figure** 2 – Distribution function of level curvatures *P*(*ln c*) in the vicinity of the critical energy Quantum Hall-to-insulator transition in the lowest Landau band for the size of the two-dimensional system *L*=200 for various disorder W (shown by symbols of different lines). Below is shown the distribution of normalized curvatures P(*ln* c*)

A sufficiently high accuracy allows us, however, to perform in addition a reliable standard one-parameter scaling procedure. From the above data one can determine in this way the scaling function for the average level curvature and the disorder dependence of the correlation length. Moreover, in the diffusive regime it has been possible to extract the weak localization corrections proportional to $e^2/(\pi h)$ (L/*l*-1), with *l* being the mean free path of an electron. We have found the critical exponent $\nu = 1.6 \pm 0.1$, which is quite close to that obtained recently by the transfer-matrix technique [18], but exceeds slightly the value found from the calculation of the level spacing distribution [19].

At present the following open questions are of particular interest:

1. How do the probability distribution functions of the level curvatures and of the level velocities behave in the crossover regime between plateaus in the IQHE?
2. What are their asymptotic forms?
3. Whether the level curvature distribution has non-analyticity at zero-curvature? If yes, how large is `the branching number' ?
4. How is the branching number linked to the multifractality spectrum?
5. How do the parametric statistics depends on the index of the Landau Level?
6. What is the velocity-velocity auto-correlation function in the center of the lowest Landau Level?
7. How does the level curvature distribution scale when moving towards the Landau band edge?

The abovementioned issues will be dealt in the out next investigations.

**Summary**

Parametric statistics of eigenvalues in the critical energy range of the Landau levels are studied. The system in the limit of widely disorder-broadened Landau bands is investigated using exact numerical diagonalization techniques for lattice models. The simulation methodology used is similar to that published in our previous work [20] devoted to the three-dimensional orthogonal case. We examine scaling properties of several statistical spectral measures at the transitions between the Hall plateaus including: the distributions of the level curvatures and scaling properties of the P(*ln* c). Using the finite-size analysis for lowest Landau levels, a set of universal constants are extracted: the branching number of the eigenvalue curvatures and the critical exponent of the localization length. Scale-invariant behavior of the parametrical statistics characteristic of the localization-delocalization transition reveals the universal nature of the integer quantum Hall effect.

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