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SOLITON SURFACE ASSOCIATED WITH THE EQUATION OF ASSOCIATIVITY FOR $N = 3$ CASE WITH AN METRIC $\eta \neq 0$

Abstract. The Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations, also called the associativity equations, is a system of nonlinear partial di□erential equations for one function, depending on a finite number of variables. The WDVV equations were introduced a few decades ago in the context of two-dimensional topological field theories.The task of giving the associativity equations a geometric interpretation has two complementary aspects. On one side, can write these equations in a form that does not depend on the choice of the coordinates. On the other side, one must demand that the geometrical structure should be capable to select a class of a□nely related coordinates. The coordinate selection rule is important in the geometrization of the associativity equations. In this paper, we consider the soliton surface of the associativity equation. The equation of associativity originated from 2D topological field theory. 2D topological field theory represent the matter sector of topological string theory. These theories covariant before coupling to gravity due to the presence of a nilpotent symmetry and are therefore often referred to as cohomological field theories. The surface is constructed using Sym-Tafel formula, which is a connection between classical manifold geometry and soliton theory.The Sym-Tafel formula reconstructs a surface from knowledge of its fundamental forms, combines integrable nonlinearities, and allows the application of soliton theory methods to geometric problems. The soliton surfaces approach is necessary in the construction of so-called integrable geometries. Any class of soliton surfaces is integrable. Geometric objects associated with the surfaces of the solitons can usually be identified with the solutions to the strings. Thus in this work soliton surfaces for the associativity equation for $n = 3$ case with an metric $\eta_{11} \neq 0$ are considered, and first and second fundamental forms of soliton surfaces are found for this case. In addition, we study an area of surfaces for the associativity equation for $n = 3$ case with an metric $\eta_{11} \neq 0$.

Key words: the equation of associativity, nonlinear equation, the Lax pair, first and second fundamental forms, soliton surfaces, area of surfaces.

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Ассоциация теңдеуімен байланысты $\eta \neq 0$ метрикасымен $n = 3$ жағдайда солитон беті

Аңдатпа. Виттен-Дижкграф-Е. Верлинде-Г. Верлинде (WDVV) теңдеулері, сондай-ақ ассоциативтілік теңдеулері деп аталатын, сызықтық емес жүйені айнымалылардың соңғы санына тәуелділігіне байланысты бір функция үшін жеке туындыдағы сызықты емес теңдеулер жүйесі болып табылады. WDVV теңдеулері бірнеше он жыл бұрын өріс теорияларының екі өлшемді топологиялық контексінде енгізілді. Геометриялық интерпретацияның өзара толық-тыратын екі аспектісі бар. Бір жағынан, бұл теңдеулерді координаттарды таңдауға байланысты емес түрде жазуға болады. Екінші жағынан, геометриялық құрылым аффинды-байланысқан координаттар класын таңдауға қабілетті болуын талап ету қажет. Координаттарды таңдау ережесі ассоциативті теңдеулерді геометриялауда маңызды рөл атқарады. Осы жұмыста ассоциативтілік теңдеуінің солитон беті қарастырылады. Ассоциативтілік теңдеуі өрістің 2D топологиялық теориясынан пайда болды. 2D топологиялық өріс теориясы ішектердің топо-логиялық теориясының материалдық секторы болып табылады. Бұл теориялар нильпотентті симметрияның болуына байланысты гравитациямен байланыстыру алдында ковариантты және сондықтан жиі өрістің когомологиялық теориялары деп аталады. Сан алуантүрлілі беттің классикалық геометриясы мен солитондар теориясының арасындағы байланыс болып табылатын Сим-Тафель формуласын қолдану арқылы құрылған. Сим-Тафель деп аталатын формула беттің іргелі формаларын білуден оның айқын қалпына келтіруін жеңілдетеді, әртүрлі интегралданбайтын бейсызықтықтарды біріктіреді және солитондар теориясының әдістерін геометриялық есептерге қолдануға мүмкіндік береді. Интегралданатын деп аталатын геометрияны құруда солитондық беттер әдісі өте пайдалы болады. Кез келген солитондық беттер класы интегралданады. Солитондық беттермен байланысқан геометриялық нысандарды ішектердің шешімдерімен анықтауға болады. Сонымен, осы мақалада $\eta_{11} \neq 0$ метрикасымен $n = 3$ жағдайы үшін WDVV теңдеуінің солитондық беттерін қарастырамыз, сонымен қатар осы жағдай үшін солитондық беттердің бірінші және екінші іргелі формалары табылған. Сонымен қатар, $\eta_{11} \neq 0$ метрикасымен $n = 3$ жағдайындағы WDVV теңдеуі үшін беттің ауданы табылған.

Түйін сөздер: ассоциативтілік теңдеуі, бейсызық теңдеу, Лакс жұбы, бірінші және екінші іргелІ пішінедр, солитон беті, беттің ауданы.

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Солитонная поверхность, связанная с уравнением ассоциативности для случая $n = 3$ с $\eta \neq 0$ метрикой

Аннотация. Уравнения Виттен-Дижкграф-Е. Верлинде-Г.Верлинде (WDVV), также называемые уравнениями ассоциативности, представляют собой систему нелинейных уравнений в частных производных для одной функции, зависящей от конечного числа переменных. Уравнения WDVV были введены несколько десятилетий назад в контексте двумерных топологических теорий поля. Задача придания уравнениям ассоциативности геометрической интерпретации имеет два взаимодополняющих аспекта. С одной стороны, можно записать эти уравнения в виде, не зависящем от выбора координат. С другой стороны, необходимо требовать, чтобы геометрическая структура была способна выбирать класс аффинно-связанных координат. Правило выбора координат играет важную роль в геометризации уравнений ассоциативности. В настоящей работе рассматривается поверхность солитона уравнения ассоциативности. Уравнение ассоциативности возникло из 2D топологической теории поля. 2D топологическая теория поля представляет собой материальный сектор топологической теории струн. Эти теории ковариантны перед связыванием с гравитацией из-за наличия нильпотентной симметрии и поэтому часто называются когомологическими теориями поля. Поверхность построена с использованием формулы Сима-Тафеля, которая является связью между классической геометрией многообразия и теорией солитонов. Формула сим-Тафеля восстанавливает поверхность из знания ее фундаментальных форм, объединяет интегрируемые нелинейности и позволяет применять методы теории солитонов к геометрическим задачам. Подход солитонных поверхностей необходим при построении так называемых интегрируемых геометрий. Любой класс солитонных поверхностей интегрируется. Геометрические объекты, связанные с поверхностями солитонов, обычно можно отождествить с решениями струн. Таким образом, в данной работе рассматриваются солитонные поверхности уравнения ассоциативности для случая $n = 3$ с $\eta_{11} \neq 0$ метрикой, а также найдены первая и вторая фундаментальные формы солитонных поверхностей для данного случая. Кроме того, найдена площадь поверхностей для уравнения ассоциативности для случая $n = 3$ с $\eta_{11} \neq 0$ метрикой.

Ключевые слова: уравнение ассоциативности, нелинейное уравнение, пара Лакса, первая и вторая фундаментальная формы, поверхность солитона, площадь поверхности.

Introduction

The WDVV relation for genus 0 Gromov-Witten (GW) invariants completely solves the classical problem of enumerating complex rational curves in the complex projective space $\mathbf{P}_n$ [1]. For genus-0 GW-theory, the associativity of quantum cohomology, which is equivalent to WDVV equation, led to Kontsevich's solution to the classical problem of counting degree d rational curves passing through 3d − 1 general points in $\mathbf{P}_2$ [2].A system of PDE, called open WDVV, that constrains the bulkdeformedsuperpotential and associated open GW invariants of a

Lagrangiansubmanifold L ⊂ X with a bounding chain [3].In this paper we shall consider so-called nonlinear partial differential equations of associativity in 2D topological field theories (see [4, 5, 6, 7]) and give their description as integrable nondiagonalizable weakly nonlinear systems of hydrodynamic type. For systems of this type corresponding general differential geometric theory of integrability connected with Poisson structures of hydrodynamic type can be developed. For an arbitrary solution of the open WDVV equations, satisfying a certain homogeneity condition, constructed a descendent potential in genus 0 [8].For any mechanics, given by the metric and the third order Codazzi tensor, it is possible to obtain the superfield Lagrangian [9] by solving a simple differential equation.Universal algebraic structure, closely related with that of the WDVV equation, govern quantum correlation functions of every quantum field theory [10].Topological approach provides a general framework for lifting relations via morphisms between not necessarily orientable spaces [11]. For isotropic (so(n)-invariant) spaces provided admissible prepotentials for any solution to the curved WDVV equations [12]. For every flat-space WDVV solution subject to a simple constraint provided a curved-space solution on any isotropic space, in terms of the rotationally invariant conformal factor of the metric [13]. Flat structure was introduced by K. Saito and his collaborators at the end of 1970's. Independently the WDVV equation arose from the 2D topological field theory. B. Dubrovin unified these two notions as Frobenius manifold structure [14].The concepts of Frobenius manifold and Lenard complex must be strictly related. They provides two ways of looking at the same object from different perspectives and by using different geometrical structures [15]. In paper [16]compared two different geometrical interpretations of the WDVV equations of 2D topological field theory. The first is the classical interpretation proposed by Boris Dubrovin, based on the concept of Frobenius manifold. The second is a novel interpretation, based on the concept of Lenard complex on a Haantjes manifold. In paper [17]. determined correlators of topological quantum field theories and provided explicit solutions to the WDVV equations.

We remind very briefly following Dubrovin [4] the basic mathematical concepts connected with the Witten-Dijkgraaf-E.Verlinde-H.Verlinde (WDVV) system arising originally in two-dimensional topological field theories [4, 5] and its relations with the Dubrovin type equations of associativity. The WDVV equations, in general, have the following form [4, 18]:

$$\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^p}\eta^{pq}\frac{\partial^3 F}{\partial t^q \partial t^k \partial t^r} =$$

$$= \frac{\partial^3 F}{\partial t^j \partial t^k \partial t^p}\eta^{pq}\frac{\partial^3 F}{\partial t^i \partial t^q \partial t^r},$$

$$\forall i, j, k, r \in \{1, ..., n\},$$

where F is a prepotential, η is a metric.

Consider a function $F(t), t = (t^1, ..., t^n)$ such that the following three conditions are satisfied for its third derivatives denoted as [4, 5]

$$c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

1) normalization, i.e.,

$$\eta_{\alpha\beta} = c_{1\alpha\beta}(t)$$

is a constant nondegenerate matrix;

2) associativity, i.e., the functions

$$c^{\gamma}_{\alpha\beta}(t) = \eta^{\gamma\varepsilon} c_{\varepsilon\alpha\beta}(t)$$

for any t define a structure of an associative algebra A_t in the n-dimensional space with a basis $e_1, ..., e_n$:

$$e_\alpha \cdot e_\beta = c^{\gamma}_{\alpha\beta}(t) e_\gamma .$$

3) $F(t)$ must be a quasihomogeneous function of its variables:

$$F(c^{d_1} t^1, ..., c^{d_n} t^n) = c^{d_F} F(t^1, ..., t^n)$$

for any nonzero c and for some numbers $d_1, ..., d_n, d_F$.

The resulting system of equations for $F(t)$ is called the Witten-Dijkgraaf-E.Verlinde-H.Verlinde (WDVV) system [6, 7] (see also[4, 5]). It was shown by Dubrovin [4] that solutions of the WDVV system can be reduced by a linear change of coordinates to two special types:

(1) in physically the most important case

$$F(t) = 12(t^1)^2 t^n + 12t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n-\alpha+1} + \\ + f(t^2, ..., t^n)$$

for some function $f(t^2, ..., t^n)$.

(2) in some special case

$$F(t) = 16(t^1)^3 + 12t^1 \sum_{\alpha=1}^{n-1} t^\alpha t^{n-\alpha+1} + \\ + f(t^2, ..., t^n).$$

In this work we consider the WDVV equations for $n = 3$ case with an metric such that $\eta_{11} \neq 0$

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Soliton surfaces for equation of associativity for $n = 3$ casewith an metric $\eta \neq 0$

For $n = 3$ case with an metric such that $\eta_{11} \neq 0$, the dependence of the function F on the fixed variable t^1 was found by Dubrovin [19, 20] which is

$$F = \frac{1}{6}(t^1)^3 + t^1 t^2 t^3 + f(t^2, t^3). \qquad (1)$$

For this case the equation of associativity reduces to the following nonlinear equation of the third order for a function $f = f(x, t)$) of two independent variables ($x = t^2, t = t^3$):

$$f_{xxx} f_{ttt} - f_{xxt} f_{xtt} = 1, \qquad (2)$$

Let us introduce new variables a, b, c as follows [20, 21]:

$$a = f_{xxx}, \quad b = f_{xxt}, \quad c = f_{xtt}. \qquad (3)$$

In the above variables the equation (2) can be rewritten as a system of three equations in the following way:

$$\begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = \left(\dfrac{(1+bc)}{a} \right)_x \end{cases} \qquad (4)$$

In the following sections we work with the system (4).

First fundamental form of a surface

The corresponding Lax pair for the WDVV equation for $n = 3$ case to the system (4) is given by

$$\Phi_x = U\Phi \qquad (5)$$

$$\Phi_t = V\Phi \qquad (6)$$

where $U = \lambda A$ and $V = \lambda B$. Here A and B matrices defined as follows [20, 21]:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & b & a \\ 1 & c & b \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & c & b \\ 0 & \dfrac{(1+bc)}{a} & c \end{pmatrix}. \qquad (7)$$

Geometrical objects associated with soliton surfaces (tangent vectors, normal vectors, foliations by curves etc.) usually can be identified with solutions to some nonlinear models [22, 23]. The scalar square of the total differential dr of the radius-vector of the current point of a surface is called the first fundamental form I of the surface [24]:

$$I = \mathrm{d}r^2, \qquad (8)$$

In expanded form, it is recorded as

$$I = r_x^2 \mathrm{d}x^2 + 2r_x r_t \mathrm{d}x\mathrm{d}t + r_t^2 \mathrm{d}t^2, \qquad (9)$$

where x and t are the curvatures.

To construct the surface, we now use the Sym-Tafel formula [25]. It has the form

$$r = \Phi^{-1}\Phi_\lambda, \qquad (10)$$

where $r = \sum r_j \sigma_j$ is the matrix form of the position vector of the surface, Φ is a solution of the equations (5)-(6). We have

$$\begin{aligned} r_x &= \Phi^{-1} U_\lambda \Phi, \\ r_t &= \Phi^{-1} V_\lambda \Phi. \end{aligned} \tag{11}$$

In terms of the Lax representation, equation (**Ошибка! Источник ссылки не найден.**) will be rewritten as follows:

$$I = \frac{1}{2}\left(\mathrm{tr}(U_\lambda^2)dx^2 + 2\mathrm{tr}(U_\lambda V_\lambda)dxdt + \mathrm{tr}(V_\lambda^2)dt^2 \right). \tag{12}$$

We now turn to finding the first fundamental form of soliton surface for the WDVV equation for n = 3 case to the system (4)

$$\mathrm{tr}(U_\lambda^2) \quad = \quad 2\left(b^2 + ac\right), \tag{13}$$

$$\mathrm{tr}(U_\lambda V_\lambda) \quad = \quad 3 + 4bc, \tag{14}$$

$$\mathrm{tr}(V_\lambda^2) \quad = \quad 2\left(c^2 + \frac{(b + b^2 c)}{a} \right) \tag{15}$$

Substituting equations (13)-(15) into equation (12) we have the first fundamental form of soliton surface for the WDVV equation to the system (4)

$$I = -\frac{1}{2}\left[2\left(b^2 + ac\right)\mathrm{d}x^2 + (3 + 4bc)\mathrm{d}x\mathrm{d}t + 2\left(c^2 + \frac{(b + b^2 c)}{a} \right)\mathrm{d}t^2 \right]. \tag{16}$$

Second fundamental form of a surface

The scalar product of the total differential of the second order $\mathrm{d}^2 r$ of the radius-vector r of the current point of a surface by the orbit of the normal n at this point is called the second quadratic form of the surface [24]:

$$II = -\mathrm{d}n \cdot \mathrm{d}r, \tag{17}$$

where

$$n = \frac{r_x \wedge r_t}{| r_x \wedge r_t |}.$$

In an expanded form, it is recorded as

$$II = b_{11} dx^2 + 2b_{12} dxdt + b_{22} dt^2, \tag{18}$$

where the coefficients b_{11}, b_{12} and b_{22} are given as

$$b_{11} \quad = \quad r_{xx} \cdot n, \tag{19}$$

$$b_{12} \quad = \quad r_{xt} \cdot n, \tag{20}$$

$$b_{22} \quad = \quad r_{tt} \cdot n, \tag{21}$$

or

$$b_{11} \quad = \quad \frac{1}{2}\mathrm{tr}(r_{xx} n), \tag{22}$$

$$b_{12} \quad = \quad \frac{1}{2}\mathrm{tr}(r_{xt} n), \tag{23}$$

$$b_{22} \quad = \quad \frac{1}{2}\mathrm{tr}(r_{tt} n), \tag{24}$$

where

$$\begin{aligned} r_{xx} &= \Phi^{-1}(U_{\lambda x} + [U_\lambda, U])\Phi, \\ r_{xt} &= \Phi^{-1}(U_{\lambda t} + [U_\lambda, V])\Phi, \\ r_{tt} &= \Phi^{-1}(V_{\lambda t} + [V_\lambda, V])\Phi \end{aligned}$$

The normal vector n is given by

$$n = \pm \frac{\Phi^{-1}[U_\lambda, V_\lambda]\Phi}{\sqrt{\frac{1}{2}\mathrm{tr}([U_\lambda, V_\lambda]^2)}}.$$

Thus, the equation (19)-(21) is written as follows

$$b_{11} = \frac{1}{2}\frac{tr\left(\left(U_{\lambda x}+[U_\lambda,U]\right)[U_\lambda,V_\lambda]\right)}{\sqrt{\frac{1}{2}tr([U_\lambda,V_\lambda]^2)}},$$

$$b_{12} = \frac{1}{2}\frac{tr\left(\left(U_{\lambda t}+[U_\lambda,V]\right)[U_\lambda,V_\lambda]\right)}{\sqrt{\frac{1}{2}tr([U_\lambda,V_\lambda]^2)}},$$

$$b_{22} = \frac{1}{2}\frac{tr\left(\left(V_{\lambda t}+[V_\lambda,V]\right)[U_\lambda,V_\lambda]\right)}{\sqrt{\frac{1}{2}tr([U_\lambda,V_\lambda]^2)}},$$

Using equation (7) we obtain that $[A,B]=0$. So, we have that $n=0$ and the second fundamental form of a soliton surface for the WDVV equation to the system (4) is

$$II = 0.$$

Area of surfaces for equation of associativity for $n=3$ case with an metric $\eta \neq 0$

In this section we consider the area of surfaces for the WDVV equation for $n=3$ to the system (4). Area of surfaces is evaluated by

$$S = \iint\sqrt{\frac{1}{2}tr\left(\{U_{\lambda x}+[U_\lambda,U]\}^2\right)}dxdt, \qquad (25)$$

where the matrix A is defined as in equation (7).

So, that $[U_\lambda,U]=0$, we have

$$(U_{\lambda x})^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_x^2+a_xc_x & 2a_xb_x \\ 0 & 2b_xc_x & b_x^2+a_xc_x \end{pmatrix}.$$

Area of surfaces (25) for the WDVV equation to the system (4) is given by

$$S = \iint\sqrt{\frac{1}{2}\left(2b_x^2+2a_xc_x\right)}\mathrm{d}x\mathrm{d}t = \\ = \iint\sqrt{b_x^2+a_xc_x}\,\mathrm{d}x\mathrm{d}t.$$

Conclusions

In this work we considered the equation of associativityfor $n=3$ case with an metric $\eta_{11} \neq 0$. Soliton surfaces for the equation of associativity equations for $n=3$ cases with an metric $\eta_{11} \neq 0$ was obtained. Area of surfaces for the equation of associativityfor $n=3$ cases with an metric $\eta_{11} \neq 0$ was investigated.

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