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APPLICATION OF GBT THEOREM FOR GRAVITATIONAL DEFLECTION OF LIGHT BY COMPACT OBJECTS

One of the most renowned classical experiments that verifies the curved nature of space-time geometry is the bending of light. This phenomenon is extensively discussed in most textbooks on general relativity, with a clear and comprehensive explanation provided.

In this study, we employ the material medium approach to determine the refractive index associated with the gravitational field of a compact object with a quadrupole moment. Our research presents a method for calculating the gravitational deflection angle for compact objects by utilizing the refractive index and the GBT theorem for an isotropic metric. This method is particularly important because it allows for the calculation of the deflection angle for both light and relativistic particles. The material medium approach enables us to consider the compact object's gravitational field as a medium with a refractive index. By applying this approach, we establish a relationship between the refractive index and the quadrupole moment of the compact object. We then utilize this relationship to calculate the deflection angle of light and relativistic particles.

Key words: GBT theorem, deflection of light, Compact Objects.

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Ықшам объектілермен жарықтың гравитациялық ауытқуы үшін Гаусс-Бонне теоремасын қолдану

Кеңістік-уақыт геометриясының қисық сипатын растайтын ең әйгілі классикалық тәжірибелердің бірі – жарықтың қисықтығы. Жарықтың ауытқуының әдеттегі түсіндірмесі қарапайым. Массивтік нысан болған кезде, жарық сәулесі әдетте әсер ету параметрі деп аталатын белгілі бір аймақтағы жүйенің жабық массасына тура пропорционал бұрышпен иіледі. Статикалық сфералық симметриялы гравитациялық өріс болған кезде жарықтың ауытқуының бірегей перспективасын ұсынатын жаңа көзқарас бар. Бұл әдіс жарық сәулелерінің траекториясының топологиясының маңыздылығын көрсетеді. Бұл әдіс ауытқу бұрышын есептеу үшін Гаусс-Бонне теоремасын қолдануды пайдаланады.

Бұл жұмыста төртполюсті моменті бар ықшам нысанның гравитациялық өрісімен байланысты сыну көрсеткішін анықтау үшін материалдық орта тәсілін қолданамыз. Біздің зерттеуіміз сыну көрсеткішін және изотропты метрика үшін Гаусс-Бонне теоремасын пайдалана отырып, жинақы объектілер үшін гравитациялық ауытқу бұрышын есептеу әдісін ұсынады. Бұл әдіс әсіресе маңызды, себебі ол жеңіл және релятивистік бөлшектер үшін ауытқу бұрышын есептеуге мүмкіндік береді. Материалдық орта тәсілі ықшам нысанның гравитациялық өрісін сыну көрсеткіші бар орта ретінде қарастыруға мүмкіндік береді. Бұл тәсілді пайдалана отырып, біз ықшам нысанның сыну көрсеткіші мен төртполюсті моменті арасындағы байланысты орнатамыз. Содан кейін біз бұл қатынасты жарық пен релятивистік бөлшектердің ауытқу бұрышын есептеу үшін пайдаланамыз.

Түйін сөздер: GBT теоремасы, жарықтың ауытқуы, ықшам нысандар.

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Применение теоремы Гаусса-Бонне для гравитационного отклонения света компактными объектами

Одним из самых известных классических экспериментов, подтверждающих искривленную природу геометрии пространства – времени, является искривление света. Обычное объяснение отклонения света простое. Когда присутствует массивный объект, световой луч изгибается под углом, который прямо пропорционален замкнутой массе системы в определенной области, обычно называемой прицельным параметром. Существует новый подход, предлагающий уникальный взгляд на отклонение света в присутствии статического сферически-симметричного гравитационного поля. Этот метод подчеркивает значение топологии траектории световых лучей. Для расчета угла отклонения этот метод использует применение теоремы Гаусса-Бонне.

В данной работе мы используем подход материальной среды для определения показателя преломления, связанного с гравитационным полем компактного объекта с квадрупольным моментом. В нашем исследовании представлен метод расчета угла гравитационного отклонения для компактных объектов с использованием показателя преломления и теоремы Гаусса-Бонне для изотропной метрики. Этот метод особенно важен, поскольку позволяет вычислить угол отклонения как для легких, так и для релятивистских частиц. Подход материальной среды позволяет рассматривать гравитационное поле компактного объекта как среду с показателем преломления. Применяя этот подход, мы устанавливаем связь между показателем преломления и квадрупольным моментом компактного объекта. Затем мы используем это соотношение для расчета угла отклонения света и релятивистских частиц.

Ключевые слова: теорема GBT, отклонение света, компактные объекты.

Introduction

The General Theory of Relativity establishes a beautiful mathematical connection between the geometry of space-time and the behavior of matter, which is described by the energy-momentum tensor. This theory has been extensively tested in the past and has provided explanations for various astrophysical phenomena. The experimental results have consistently supported the theoretical predictions, indicating a remarkable agreement.

One of the most renowned classical experiments that verifies the curved nature of space-time geometry is the bending of light. This phenomenon is extensively discussed in most textbooks on general relativity, with a clear and comprehensive explanation provided. The conventional explanation for the deflection of light is straightforward. When a massive object is present, the light ray is bent by an angle that is directly proportional to the enclosed mass of the system within a specific area, commonly referred to as the impact parameter [1]. Gibbons and Werner identified a novel approach that offers a unique perspective on the deflection of light in the presence of a static, spherically symmetric gravitational field. This method underscores the significance of topology on the trajectory of light rays [1,2]. To calculate the deflection angle, this technique employs the application of the Gauss-Bonnet theorem (GBT).

There are several solutions of Einstein field equations that can be used to describe the exterior gravitational field of a static mass distribution with quadrupole moment [3]. In the limiting case of vanishing quadrupole, they reduce to the space time of a Schwarzschild black hole. A common feature of these metrics is that in all of them, the hypersurface $r = 2m$ is singular.

Thus, further we apply the Gauss-Bonnet theorem for slightly deformed compact objects and show that in this case we get the same result as in the standard approach.

Methods. The metric of slightly deformed compact object

There are several solutions of Einstein field equations that can be used to describe the exterior gravitational field of a static mass distribution with quadrupole moment [3]. In the limiting case of vanishing quadrupole, they reduce to the space time of a Schwarzschild black hole. A common feature of these metrics is that in all of them, the hypersurface r = 2m is singular. Recently, in [4], a different approximate generalization of the Schwarzschild metric with a quadrupole was derived, which can be written as

$$ds^2 = \left(1-\frac{2m}{r}\right)\left(1-\frac{qm}{r-m}\right)dt^2 - \left[1+\frac{qm(r-2m)}{(r-2m)^2}\right]\frac{dr^2}{1-\frac{2m}{r}} - \left(1+\frac{qm}{r-m}\right)r^2(d\theta^2+\sin^2\theta\, d\varphi^2) \quad (1)$$

where m is the mass and q is the quadrupole parameter. The most interesting feature of this approximate solution is that its Kretschmann scalar

$$K = \frac{48m^2}{r^6}\left(1+\frac{qr-4m}{r-m}\right) + O(q^2), \quad (2)$$

shows that the hypersurface $r = 2m$ is regular. Instead, there exists a curvature singularity at $r = m$ aside from the central singularity located at $r = 0$, which is also present in the Schwarzschild spacetime.

To further analyze the physical meaning of the solution (1), we calculate the corresponding Newtonian limit. To this end, we perform a coordinate transformation of the form $(r,\theta) \to (R,\vartheta)$ defined by the equations [4-6]

$$r = R\left[1 - \frac{qm}{R\left[1+\frac{m}{R}(\beta_1+\sin^2\vartheta)+\frac{m^2}{R^2}(\beta_2+\sin^2\vartheta)+..\right]\sin^2\vartheta}\right], \quad (3)$$

and

$$\theta = \vartheta - q\frac{m^2}{R^2}\left(1+2\frac{m}{R}+..\right) sin\vartheta cos\vartheta, \quad (4)$$

where the β_1 and β_2 are constants and we have neglected terms of the order higher that m^3/R^3. Inserting the above coordinates into the metric (1), we obtain the approximate line element

$$ds^2 = (1+2\Phi)dt^2 - \frac{dR^2}{1+2\Phi} - U(R,\vartheta)R^2(d\vartheta^2+\sin^2\vartheta d\varphi^2), \quad (5)$$

with

$$\Phi = -\frac{GM}{R} + \frac{GQ}{R^3}P_2(cos\vartheta), \quad (6)$$

$$U(R,\vartheta) = 1 - \frac{2GM}{R^3}P_2(cos\vartheta), \quad (7)$$

where $P_2(cos\,\theta)$ is the Legendre polynomial of degree 2, and we have chosen the free constants as $\alpha_2 = 2, \beta_1 = 1/3$, and $\beta_2 = 5/3$.

We recognize the metric (5) as the Newtonian limit of general relativity, where Φ represents the Newtonian potential. Moreover, the constants

$$M = (1+q)m, Q = \frac{2}{3}qm^3 \quad (8)$$

can be interpreted as the Newtonian mass and quadrupole moment of the corresponding mass distribution.

Representation of metric in the isotropic coordinates and calculation of refractive index

Writing the metric tensor in isotropic coordinates is a convenient and practical approach for calculating the refractive index in curved spacetime, as it simplifies the mathematical calculations involved and makes the problem more tractable. For the calculation of the corresponding refractive index, we now represent the above approximate metric in isotropic coordinates (t,ρ,θ,φ). To this end, let us consider the coordinate transformation

$$r = \rho\left(1+\frac{m}{2\rho}\right)^2 + qh(\rho), \quad (9)$$

where the additional auxiliary function $h(\rho)$ stays for the deformation from spherical symmetry. By introducing this transformation into the line element in Equation (1) and expanding up to the first order in q, the corresponding function h (ρ) turns out to be

$$h(\rho) = \frac{m^2}{4\rho}\frac{(m^2-4\rho^2)}{(m^2+4\rho^2)}. \quad (10)$$

Finally, we can rewrite the line element in the form

$$ds^2 = A(\rho)dt^2 - B(\rho)d\vec{\rho}^2, \qquad (11)$$

where we have introduced the notation

$$d\vec{\rho}^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta \, d\varphi^2, \qquad (12)$$

and the functions $A(\rho)$ and $B(\rho)$ are

$$A(\rho) = \frac{(m-2\rho)^2}{(m+2\rho)^2}\left[1 + q\frac{4m\rho}{m^2-4\rho^2}\right], \qquad (13)$$

$$B(\rho) = \frac{(m+2\rho)^4}{16\rho^4}\left[1 + q\frac{2m}{m+\rho}\right]. \qquad (14)$$

In the limiting case $q = 0$, the above metric reduces to the Schwarzschild metric in the isotropic form [7]. To derive the expression for the refractive index, we follow the procedure proposed by Sen in [7] for static fields. We consider the trajectory of a light ray such that $ds^2 = 0$. Then, the velocity of the light ray $v(\rho)$ can be determined from the expression

$$v^2(\rho) = \frac{d\vec{\rho}^2}{dt^2} = \frac{A(\rho)}{B(\rho)} = \frac{16(m-2\rho)^2\left(1+\frac{4qm\rho}{m^2-4\rho^2}\right)\rho^4}{(m+2\rho)^6\left(1+\frac{2qm}{m+\rho}\right)}. \qquad (15)$$

Using $n(\rho) = c/v(\rho)$ the last equation yields the effective refractive index for light in the gravitational field

$$n(\rho) = \frac{1}{4\sqrt{\frac{\rho^4(m-2\rho)^2}{(m+2\rho)^6}}} + \frac{1}{4}\frac{(m^2-2m\rho-6\rho^2)m}{\sqrt{\frac{\rho^4(m-2\rho)^2}{(m+2\rho)^6}}(m^2-4\rho^2)(m+\rho)}q + O(q^2). \qquad (16)$$

In this way the optical metric reads

$$dt^2 = n(\rho)^2 d\rho^2 + \rho^2 n^2(\rho) d\varphi^2. \qquad (17)$$

We will use this expression further at calculation deflection angle.

Furthermore, the velocity of the light ray in terms of the radial coordinate r can be obtained as follows:

$$v(r) = v(\rho)\left(\frac{dr}{d\rho}\right), \qquad (18)$$

where the expression $dr/d\rho$ can be determined from Equations (9) and (10) as follows:

$$\frac{dr}{d\rho} = 1 - \frac{m^2}{4\rho^2} + \frac{qm^2}{4}\frac{(16m^2\rho^2 - m^4)}{\rho^2(m^2+4\rho^2)^2}. \qquad (19)$$

Using the inverse transformation, the new coordinate ρ can be written as

$$\rho = \frac{1}{2}(2r-3m) + \frac{qm^2}{4(r-m)}. \qquad (20)$$

Finally, the refractive index is related to the velocity by

$$n(r) = \frac{1}{v(r)}. \qquad (21)$$

Then, from Equations (18), (19), and (21), we obtain

$$n(r) = \frac{r}{r-2m}\left[1 + \frac{q}{2}\frac{m(2r-3m)}{(r-m)^2}\right], \qquad (22)$$

this is the effective refractive index of the spacetime described by the approximate quadrupolar metric in Equation (1). As expected, in the limiting case $q = 0$, it reduces to refractive index of the Schwarzschild spacetime [7], where $n_0 = r/(r-2m)$. The second term is due to the slight deformation of the central gravitating body from the spherical symmetry. The behavior of the refractive index is illustrated in Figure 1. Notice that the divergences located at $r = m$ and $r = 2m$ are due to the presence of the curvature singularity of the metric in Equation (1) and the Schwarzschild horizon, respectively.

In particular, in the weak field approximation, the refractive index can be expressed by the infinite converging series

$$n(r)_{weak} \approx 1 + \frac{2m}{r} + q\frac{m}{r} + \qquad (23)$$

In the case of vanishing quadrupole parameter, the above expression yields the refractive index derived by different authors for the weak field limit of the Schwarzschild metric [8-10].

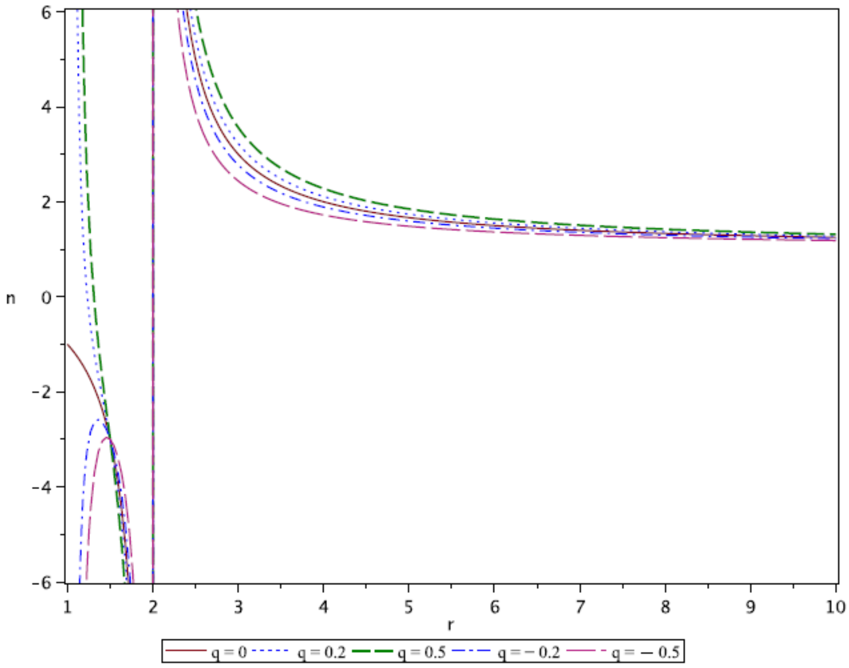

Figure 1. Refractive index as a function of r for different values of q and $m = 1$. In the interval $r \in (1, 2)$, the refractive index can become negative, depending on the value of q.

The GBT theorem and deflection of light

First, let's express the optical metric (17) using a set of new coordinates, which includes the introduction below

$$r^* = n(\rho)d\rho, \quad (24)$$

and

$$f(r^*) = n(\rho)\rho. \quad (25)$$

At this moment, the optical metric indicates .

$$dt^2 = \tilde{g}_{ab}dx^a dx^b = dr^{*2} + f^2(r^*)d\varphi^2. \quad (26)$$

The expression for the Gaussian optical curvature κ is given in coordinates as

$$\kappa = -\frac{1}{f(r^*)}\frac{d^2 f(r^*)}{dr^{*2}} = -\frac{1}{f(r^*)}\left[\frac{d\rho}{dr^*}\frac{d}{d\rho}\left(\frac{d\rho}{dr^*}\right) + \frac{d\rho}{dr^*}\frac{d^2 f}{d\rho^2}\right]. \quad (27)$$

Furthermore, we can express the last equation in terms of the refraction index resulting with

$$\kappa = \frac{n(\rho)n''(\rho)\rho - \left(n'(\rho)\right)^2\rho + n(\rho)n'(\rho)}{n^4(\rho)\rho}. \quad (28)$$

Using the expression (28) and refractive index we obtain Gaussian optical curvature in weak field as above

$$\kappa \approx \left(\frac{2}{r^3} + \frac{q}{r^3}\right)M + O(M^2, q^2). \quad (29)$$

A GBT theorem allows computation the deflection angle by solving the integral bellow, a more detailed description can be found in work[1], and it is

$$\theta = -\int_0^{\pi}\int_{r_\gamma}^{\infty} \kappa\sqrt{\tilde{g}}dr^* d\varphi. \quad (30)$$

Substituting (29) into (30) we get

$$\theta = -\int_0^{\pi}\int_{\frac{b}{sin\varphi}}^{\infty} \left(\left(\frac{2}{r^3} + \frac{q}{r^3}\right)M\right)\sqrt{\tilde{g}}dr^* d\varphi, \quad (31)$$

where we have used the light ray equation

$$r_\gamma = \frac{b}{sin\varphi}. \quad (32)$$

Note that we have also used

$$dS = \sqrt{\tilde{g}}dr^*d\varphi = n^2(\rho)\rho d\rho d\varphi \approx rdrd. \quad (33)$$

Solving the last integral, we find the total deflection angle

$$\theta = \frac{4M}{b} + \frac{2q}{b} = \frac{4M}{b}\left(1 + \frac{q}{2}\right). \quad (34)$$

We also note that the same result we obtained by using standard approach of computing deflection angle due to solving integral

$$\theta = 2\int_b^\infty \frac{dr}{r\sqrt{\left(\frac{n(r)r}{n(b)b}\right)^2 - 1}} - \pi. \quad (35)$$

which has been solved by corresponding linearized expression for the deflection angle [8,11-15].

Conclusion

To summarize, this study applied the material medium approach to investigate the bending of light in the gravitational field of a mass distribution with a quadrupole. The approach treated the bending of light as a refraction effect and offered several advantages in analyzing the phenomenon. The study proposed a novel method for calculating the gravitational deflection angle for compact objects based on the refractive index and the geometric-optics approximation in an isotropic metric.

We have applied a new approach to calculating the gravitational deflection angle for compact objects based on the refractive index and GBT for an isotropic type of metric. The proposed method can determine the deflection angle of both light and relativistic particles, which is of significant importance in astrophysics and cosmology. The study provides a new perspective on gravitational lensing and can contribute to a better understanding of the effects of gravity on light and particles.

Future research could extend this approach to more complex and realistic astrophysical scenarios, such as studying the gravitational lensing effect of dark matter or incorporating the effects of black holes' spin. Additionally, the proposed method could be applied to analyze the data from ongoing and future observational campaigns, such as the Event Horizon Telescope or the upcoming Euclid mission. The material medium approach and the GBT method for calculating gravitational deflection angles can offer valuable insights into the fundamental properties of gravity and its effects on light and particles. These findings could have far-reaching implications for our understanding of the universe's structure and evolution.

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