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THE INFLUENCE OF DEFORMATION IN COMPACT OBJECTS ON REDSHIFT AND RADAR ECHO DELAY

The analogy between optics and mechanics not only provides a cohesive framework to understand particle motion in mechanics and light propagation in geometrical optics, but it also extends to the intricate realm of general relativity, especially when considering static metrics. Recent scholarly publications have fortified the notion that such an analogy is remarkably applicable to general relativity's complex scenarios. In the research presented in this paper, we delve into the material medium approach, which allows us to deduce the refractive index correlated with the gravitational field emanating from a compact celestial object endowed with a quadrupole moment. Exploring within the confines of this approach, we scrutinize the influence of the quadrupole parameter on the frequency modulation of photons, adhering strictly to the relativistic approximation parameters. This scrutiny leads us to a deeper investigation of the renowned redshift phenomenon, a pivotal concept in the general theory of relativity. Furthermore, leveraging the calculated refractive index, we probe into the nuances of radar echo delay, conscientiously accounting for the quadrupole moment's impact. The findings may offer insights that could potentially be useful in further theoretical investigations of GR.

Keywords: compact object, refractive index, gravitational redshift, radar echo delay, material mediumapproach.

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Влияние деформации компактных объектов на красное смещение и задержку радиолокационного эха

Аналогия между оптикой и механикой не только обеспечивает целостную структуру для понимания движения частиц в механике и распространения света в геометрической оптике, но и распространяется на запутанную область общей относительности, особенно при рассмотрении статических метрик. Недавние научные публикации укрепили представление о том, что такая аналогия удивительно применима к сложным случаям общей теории относительности. В исследовании, представленном в этой статье, мы рассматриваем подход материальной среды, который позволяет нам вывести показатель преломления, коррелирующий с гравитационным полем, исходящим от компактного небесного объекта, наделенного квадрупольным моментом. Исследуя в рамках этого подхода, мы тщательно изучаем влияние квадрупольного параметра на частотную модуляцию фотонов, строго придерживаясь параметров релятивистского приближения. Этот анализ приводит нас к более глубокому исследованию известного явления красного смещения - ключевой концепции общей теории относительности. Кроме того, используя вычисленный коэффициент преломления, мы исследуем нюансы задержки радиолокационного эха, учитывая влияние квадрупольного момента. Полученные результаты могут дать представление о том, что потенциально может быть полезно в дальнейших теоретических исследованиях ОТО.

Ключевые слова: компактный объект, показатель преломления, гравитационное красное смещение, задержка радиолокационного эха, подход материальной среды.

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Ықшам оъектілерде деформацияның қызыл ығысу мен радар эхо кешігуіне әсері

Оптика мен механика арасындағы үйлестік механикадағы бөлшектердің қозғалысын және геометриялық оптикадағы жарықтың таралуын түсіну үшін біртұтас негізді қамтамасыз етіп қана қоймай, сонымен қатар ол жалпы салыстырмалылықтың күрделі саласына, әсіресе статикалық метриканы қарастырғанда таралады. Соңғы ғылыми жұмыстар мұндай ұқсастық жалпы салыстырмалық теориясының күрделі жағдайларына қолдану өте қолайлы деген түсінікті бекітті. Осы жұмыста ұсынылған зерттеуде біз квадроупольді моменті бар ықшам аспан объектісінен шығатын гравитациялық өріспен корреляциялық сыну көрсеткішін шығаруға мүмкіндік беретін материалдық орта тәсілін қарастырамыз. Осы тәсілдің шеңберінде зерттей отырып, біз релятивистік жуықтау параметрлерін қатаң сақтай отыра, фотондардың жиілік модуляциясына квадрупольдік параметрдің әсерін мұқият зерттейміз. Бұл зерттеу жалпы салыстырмалылық теориясындағы негізгі тұжырымдамасы болып табылатын әйгілі қызыл ығысу құбылысын тереңірек зерттеуге көмектеседі. Сонымен қатар квадроупольді моменттің әсерін саналы түрде есептей отырып, радар эхо кешігуінің ерекшеліктерін есептелген сыну көрсеткішін қолдана отырып зерттейміз. Алынған нәтижелер жалпы салыстырмалы теориясының одан әрі теориялық зерттеулерінде пайдалы болуы мүмкін түсініктерді ұсына алады.

Түйін сөздер: ықшам объект, сыну көрсеткіші, гравитациялық қызыл ығысу, радиолокациялық эхо кешігуі, материалдық ортаға жақындау.

Introduction

One well-known and trustworthy method for investigating gravitational effects is the material medium approach, initially employed by Tamm [1] and subsequently utilized by Balazs [2] to compute how a rotating object influences the polarization of light. In this approach, the gravitational influence on the path of light is simplified into the problem of wave propagation within a material medium within a flat spacetime. This principle forms the foundation of the material medium approach. This technique is appealing since it implies that classical optics is just as valid as Riemannian geometry when examining electromagnetic effects within a weak gravitational field.

Gravitational redshift is one of the consequences of the general theory of relativity [3], but its origin predates the development of this theory. Just three years after Einstein created the special theory of relativity, he predicted the effect of gravitational redshift [4], which was due to the equivalence principle [4, 5]. Thus, testing the gravitational redshift is also considered as testing the equivalence principle [6]. The demonstration of determining the gravitational redshift from the refractive index can be found in [7, 8].

Radar echo delay in the context of general relativity (GR), this phenomenon refers to the influence of the gravitational field on the propagation of electromagnetic signals such as radar waves. In the presence of a gravitational field (such as near a planet or star), radar waves are bent and slowed down, causing a delay in receiving an echo. Thus, in the context of general relativity, the delay of a radar echo is due to the influence of gravity on space-time and, therefore, on the speed and path of propagation of radar waves.

In Einstein's theory, when dealing with a vacuum, a gravitational field that possesses spherical symmetry is mathematically represented by the Schwarzschild solution. According to Birkhoff's theorem, this solution is considered unique. When deviations from this spherical symmetry occur, they are typically described using multipole moments, with the quadrupole moment being the most significant. In situations involving an axially symmetric mass distribution with a quadrupole moment, there is no theorem guaranteeing uniqueness, and as a result, various metrics can be used to describe the corresponding gravitational field [9–12].

In fact, compact objects deviate from spherical symmetry and have their own rotation, and the effect of this deformation on redshift and radar echo delay is often neglected. In this work, we study the contribution of deformation to relativistic redshift effect, radar echo delay within the framework of the material medium approach. The deviation from the spherical symmetry is described by a quadrupole parameter [13 – 15].

Methods

Refractive index for static deformed objects

A specific straightforward metric was introduced in [16]. This metric suggested using the Zipoy–Voorhees transformation [17, 18] to create a quadrupolar vacuum solution. This metric is commonly referred to in scholarly works as the Zipoy-Voorhees metric, δ-metric, γ-metric, and q-metric [17–24]. Interior quadrupole solutions were discovered in [25], and a technique for generating perfect-fluid quadrupolar solutions was outlined in [26]. Preliminary interior solutions and characteristics of the outer q-metric were discussed in [9, 27, 28]. Recently, in [29], six varied adaptations of the Schwarzschild metric with quadrupole were examined. A notable trait of these metrics is that the hypersurface $r = 2m$ is always singular. It's possible that other precise solutions to Einstein's equations in a vacuum share these characteristics.

In a recent study presented in [9], an alternative approximate extension of the Schwarzschild metric incorporating quadrupole was formulated. This can be expressed as:

$$ds^2 = \left(1 - \frac{2m}{r}\right)\left(1 - \frac{qm}{r-m}\right)dt^2 - -\left[1 + \frac{qm(r-2m)}{(r-m)^2}\right]\frac{dr^2}{1 - \frac{2m}{r}} - -\left(1 + \frac{qm}{r-m}\right)r^2(d\theta^2 + sin^2\theta d\varphi^2) \quad (1)$$

where in m represents the mass and q denotes the quadrupole parameter.

Conversely, a curvature singularity emerges at $r = m$, in addition to the central singularity situated at $r = 0$, a feature also observed in the Schwarzschild spacetime. To our current understanding, this metric represents the sole instance with a quadrupole devoid ofsingularities at $r = 2m$. This unique characteristic alone warrants a deeper investigation into this metric.

To further analyze the physical meaning of the solution (1), we calculate the corresponding Newtonian limit. To this end, we perform a coordinate transformation of the form $(r, \theta) \to (R, \upsilon)$ defined by the equations [30, 31].

$$r = R\left[1 - q\frac{m}{R}\left[1 + \frac{m}{R}(\beta_1 + sin^2\upsilon) + + \frac{m^2}{R^2}(\beta_2 - sin^2\upsilon) + \cdots\right]sin^2\upsilon\right] \quad (2)$$

and

$$\theta = \upsilon - q\frac{m^2}{R^2}\left(1 + 2\frac{m}{R} + \cdots\right)\sin\upsilon\ cos\upsilon \quad (3)$$

where the β_1 and β_2 are constants and we have neglected terms of the order higher that m^3/R^3. Inserting the above coordinates into the metric (1), we obtain the approximate line element

$$ds^2 = (1 + 2\Phi)dt^2 - \frac{dR^2}{1 + 2\Phi} - -U(R,\upsilon)R^2(d\upsilon^2 + sin^2\upsilon d\varphi^2) \quad (4)$$

with

$$\Phi = -\frac{GM}{R} + \frac{GQ}{R^3}P_2(cos\upsilon), \quad (5)$$

$$U(R,\upsilon) = 1 - 2\frac{GM}{R^3}P_2(cos\upsilon), \quad (6)$$

where $P_2(cos\upsilon)$ is the Legendre polynomial of degree 2, and we have chosen the free constantsas $\alpha_2 = -2$, $\beta_1 = 1/3$, and $\beta_2 = 5/3$. We recognize the metric (4) as the Newtonian limit of general relativity, where Φ represents the Newtonian potential. Moreover, the constants

$$M = (1 + q)m, Q = \frac{2}{3}qm^2 \quad (7)$$

can be interpreted as the Newtonian mass and quadrupole moment of the corresponding mass distribution.

To compute the associated refractive index, we shall express the aforementioned approximate metric in isotropic coordinates denoted as $(t, \rho, \theta, \varphi)$. In pursuit of this objective, we shall delve into the coordinate transformation:

$$r = \rho\left(1 + \frac{m}{2\rho}\right) + qh(\rho). \quad (8)$$

In which the supplementary auxiliary function $h(\rho)$ represents the deviation from spherical symmetry. By incorporating this transformation into the line element denoted by (1) and approximating to the first order with respect to q, the ensuing function $h(\rho)$ can be articulated as:

$$h(\rho) = \frac{m^2(m^2 - 4\rho^2)}{4\rho(m^2 + 4\rho^2)}. \quad (9)$$

Ultimately, the line element can be reformulated in the following manner:

$$ds^2 = \frac{(m-2\rho)^2}{(m+2\rho)^2}\left[1 + q\frac{4m\rho}{m^2 - 4\rho^2}\right]dt^2 - \\ -\frac{1}{16}\frac{(m+2\rho)^4}{\rho^4}\left[1 + q\frac{2m}{m+\rho}\right]d\vec{\rho}^2. \quad (10)$$

We have introduced a notation as follows:

$$d\vec{\rho}^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin\theta^2\, d\varphi^2, \quad (11)$$

In the special scenario where $q = 0$, the aforementioned metric converges to the Schwarzschild metric presented in its isotropic representation [32, 33] Introduced by Senin [33] applicable to static fields. Evaluating the path of a light ray under the condition $ds^2 = 0$, the velocity $\upsilon(\rho)$ of the light ray can be deduced from the subsequent expression:

$$\upsilon^2(\rho) = \frac{d\vec{\rho}^2}{dt^2}. \quad (12)$$

Additionally, the velocity of the light ray, when articulated in terms of the radial coordinate r, can be derived as:

$$\upsilon(r) = \upsilon(\rho)\left(\frac{dr}{d\rho}\right), \quad (13)$$

wherein the term dr/dρ can be deduced from the given equations. (8) and (9) as

$$\frac{dr}{d\rho} = 1 - \frac{m^2}{4\rho^2} + \frac{qm^2}{4}\frac{(16m^2\rho^2 - m^4)}{\rho^2(m^2 + 4\rho^2)^2}. \quad (14)$$

Employing the inverse transformation, the novel coordinate ρ can be expressed in the form:

$$\rho = \frac{1}{2}(2r - 3m) + \frac{qm^2}{4(r-m)}. \quad (15)$$

Conclusively, the refractive index bears a relationship with the velocity, expressed as:

$$n(r) = \frac{1}{\upsilon(r)}. \quad (16)$$

Then, from Eqs.13, 14, and 16 we take

$$n(r) = \frac{r}{r-2m}\left[1 + \frac{q}{2}\frac{m(2r-3m)}{(r-m)^2}\right]. \quad (17)$$

Specifically, within the weak field approximation, the refractive index can be delineatedusing an endlessly converging series.

$$n(r)_{weak} \approx 1 + \frac{2m}{r} + q\frac{m}{r} + \cdots. \quad (18)$$

For scenarios where the quadrupole parameter approaches zero, the aforementioned expression aligns with the refractive index postulated by various scholars in the context of the weak field approximation for the Schwarzschild metric, as referenced in [33–35].

Gravitational redshift

Gravitational redshift is conceptualized as an alteration in the photon's frequency corresponding to a change in the intensity of the gravitational field it occupies. Specifically, asthe gravitational field's strength diminishes, the photon's frequency concurrently decreases, resulting in a redshift.

We restore all normalized units within the framework of the General Theory of Relativity, the equation delineating this frequency shift is presented as:

$$\omega = \frac{\omega_0}{g_{00}} \approx \omega_0\left(1 + \frac{2\gamma m}{rc^2}\right). \quad (19)$$

In this context, ω_0 denotes the consistent frequency of the photon, gauged in world time, which remains unaltered as the light ray advances. Meanwhile, ω symbolizes the frequencyof that same photon, yet gauged in proper time, displaying variability across distinct spatial points. For instance, when a photon is emitted from a massive star, its frequency near thestar at reduced r values is observed to exceed that at greater r values distant from the star.At the asymptotic boundary, within a flat space domain devoid of gravitational influence, world time aligns with proper time, rendering ω_0 as the discernible photon frequency. Now, let's approach the issue from the perspective of the material medium.

Under these circumstances, the photon's frequency within the field and its frequency within the vacuum are interconnected through a specific relationship.

$$\omega = \omega_0 \cosh\theta = \omega_0 n. \qquad (20)$$

Upon substituting Equation 18 into Equation 20, we obtain the following expression:

$$\omega = \omega_0 n = \omega_0 \left(1 + \frac{2\gamma m}{rc^2} + q\frac{\gamma m}{rc^2}\right). \qquad (21)$$

In limiting cases, when the quadrupole parameter is set to zero, the above expression reduces to the expression for a spherically symmetric gravitational source.

Radar echo delay

The radar echo delay formula, as per the material medium approach, is based on Fermat's principle of least time, which takes into account the influence of a varying refractive index *n(r).* This principle posits that light (or radar signals) follow the path that minimizes the time required to travel between two points.

$$\Delta t = 2\left(\int_{R_1}^{R_2} \frac{n(r)}{c}dr - \int_{R_1}^{R_2} \frac{1}{c}dr\right). \qquad (22)$$

The expression (22), represents the time it takes for the radar signal to travel through themedium with a varying refractive index $n(r)$ from the point of transmission R_1 to the point of reception R_2. This integral accounts for the effects of the medium's refractive index. The second integral, represents the time it would take for the radar signal to travel the same pathin a vacuum (i.e., without any medium or refractive effects) from R_1 to R_2. This formula reflects the difference in travel times between the actual path through the medium with refractive effects and the hypothetical path in a vacuum. It considers the influenceof the medium's refractive index on the radar signal's propagation and echo delay in the presence of gravity. Substituting (18) into (22) equations and restoring normalized units we obtain the result.

$$\Delta t = \frac{2\gamma m}{c^2}\ln\left(\frac{R_2}{R_1}\right)^{2+q}. \qquad (23)$$

In specific scenarios, when $q = 0$, expression (23) coincides with the formula obtained in previous works [36–38].

Results and Discussion

The formula represents the effective refractive index corresponding to the spacetime governed by the approximate quadrupolar metric (1). As anticipated, when $q = 0$, this metric converges to the refractive index associated with the Schwarzschild spacetime as indicated in [32], given by $n_0 = r/(r - 2m)$. The subsequent term arises from the minor deviation of the central gravitational entity from perfect spherical symmetry. It is noteworthy to mention that the divergences found at $r = m$ and $r = 2m$ stem from the curvature singularity of the metric (1) and the Schwarzschild horizon, respectively.

The results (17), (21) and (23) show that the quadrupole parameter affects the photon frequency the redshift and the radar echo delay within the relativistic approximation. In specific scenarios, when $q = 0$, all obtained expressions coincide with the formula obtained in the case of spherically symmetric cases [36–38].

Conclusion

In summary, we applied a material medium approach to investigate the effects of general relativity such as redshift and radar echo delay in the gravitational field with quadrupole mass distribution. As we can see, the effects under consideration completely coincide withthe results obtained for the Schwarzschild field when $q = 0$. The advantage of this methodis its intuitive clarity and ease of calculation. Despite the fact that applying our results to objects like the Earth and the Sun may seem less relevant due to their relatively weak deformations and gravitational fields, they can be very useful during the study of compact objects such as neutron stars, white dwarfs and black holes. This is particularly true withthe advent of unique sky research instruments like the Event Horizon Telescope. However,for a more general case, these effects also should be considered in the off equatorial plane.

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