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WEAK GRAVITATIONAL LENSING OF BLACK HOLES IN GENERALIZED COSMOLOGICAL MODELS WITH K-ESSENCE

This article considers the extension of the general theory of relativity by introducing additional geometric invariants in the action, which leads to a modified black hole metric. Based on the model with a metric-affine approach, an analytical analysis of gravitational lensing and the formation of a black hole shadow in a plasma medium is performed. The influence of weak gravitational lensing of black holes is investigated within the framework of cosmological models extended by scalar fields of k-essence. Models with k-essence, nonlinear kinetic dynamics, are attractive candidates for explaining the accelerated expansion of the universe without introducing a cosmological constant. Detailed formulas are provided that describe the structure of the horizon, the photon sphere, the angle of light deviation, and the lensing equations. Within such models, the deformation of the image of light sources passing near black holes was analyzed, taking into account the change in the effective metric. The modifications of the deflection angle and the distribution of shifts arising due to weak lensing were considered. The results obtained show that k-essence fields can lead to observable corrections to the standard types of general relativity, which opens up new prospects for using gravitational lensing as a tool for testing the fundamental properties of dark energy and modified gravity in astrophysical observations.

Keywords: gravitational lensing, shadow of a black hole, plasma medium, modified metric, weak gravitational field.

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К-мәні бар жалпыланған космологиялық модельдердегі қара құрымның әлсіз гравитациялық линзасы

Бұл мақалада қара тесіктің өзгертілген метрикасына әкелетін қосымша геометриялық инварианттарды енгізу арқылы жалпы салыстырмалылықтың кеңеюі қарастырылады. Метрикалық-аффиндік тәсілмен модель негізінде гравитациялық линзалау мен плазмалық ортадағы қара тесіктің көлеңкесін қалыптастыруға аналитикалық талдау жүргізілді. k-эссенция скалярлық өрістерімен кеңейтілген космологиялық модельдер шеңберіндегі қара саңылаулардың әлсіз гравитациялық линзалауының әсері зерттеледі. k-эссенция модельдері, сызықтық емес кинетикалық динамика, космологиялық тұрақтыны енгізбестен ғаламның жеделдетілген кеңеюін түсіндіруге тартымды үміткерлер болып табылады. Горизонт құрылымын, Фотон сферасын, жарықтың ауытқу бұрышын және линзалау теңдеулерін сипаттайтын егжей-тегжейлі формулалар берілген. Осындай модельдер шеңберінде тиімді метриканың өзгеруін ескере отырып, қара тесіктерге жақын орналасқан жарық көздерінің кескінінің деформациясы талданды. Әлсіз линзалау нәтижесінде пайда болатын ауытқу бұрышы мен распределысу үлестірімінің модификациялары қарастырылады. Нәтижелер k-маңызды өрістер жалпы салыстырмалылықтың стандартты түрлеріне бақыланатын түзетулерге әкелуі мүмкін екенін көрсетеді, бұл гравитациялық линзалауды астрофизикалық бақылаулардағы қараңғы энергия мен модификацияланған ауырлық күшінің негізгі қасиеттерін сынау құралы ретінде пайдаланудың жаңа перспективаларын ашады.

Түйін сөздер: гравитациялық линзалау, қара тесіктің көлеңкесі, плазмалық орта, өзгертілген метрика, әлсіз гравитациялық өріс.

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Слабое гравитационное линзирование черных дыр в обобщенных космологических моделях с к-эссенцией

В этой статье рассматривается расширение общей теории относительности путем введения дополнительных геометрических инвариантов в действие, что приводит к модифицированной метрике черной дыры. На основе модели с метрико-аффинным подходом проведен аналитический анализ гравитационного линзирования и формирования тени черной дыры в плазменной среде. Исследуется влияние слабого гравитационного линзирования черных дыр в рамках космологических моделей, расширенных за счет скалярных полей k-эссенции. Модели k-эссенцией, нелинейной кинетической динамикой, являются привлекательными кандидатами для объяснения ускоренного расширения Вселенной без введения космологической постоянной. Приведены подробные формулы, описывающие структуру горизонта, сферу фотонов, угол отклонения света и обладающие равнения линзирования. В рамках таких моделей была проанализирована деформация изображения источников света, проходящего в близи черных дыр, с учетом изменения эффективной метрики. Рассмотрены модификации угла отклонения и распределения сдвигов возникающих из-за слабого линзирования. Полученные результаты показывают, что k-эссенциальных полей может приводить к наблюдаемым поправкам к стандартным видам общей теории относительности, это открывает новые перспективы в использовании гравитационного линзирования как инструмента для тестирования фундаментальных свойств темной энергии и модиффицированной гравитации в астрофизических наблюдениях.

Ключевые слова: гравитационное линзирование, тень черной дыры, плазменная среда, модифицированная метрика, слабое гравитационное поле.

Introduction

An extension of the classical General Theory of Relativity (GTR) becomes necessary to explain modern observational data (dark energy, dark matter, inflation). The traditional model of GR based on the Einstein-Hilbert $S_{EH} = \frac{1}{16\pi}\int d^4x\sqrt{-g}R$ action successfully describes gravitational phenomena in a wide range of scales but faces difficulties in interpreting cosmological observations. To solve these problems modified theories of gravitation have been proposed. In this context, we have previously studied extended gravity $F(R,Q)$, which considers not only the curvature scalar R but also the non-metricity scalar Q. Considering the problems of classical GR, such as the necessity to explain the dark energy and inflation, it is logical to generalize the model to the functional $S = \int d^4x\sqrt{-g}F(R,Q,X,\phi)$, where X is the kinetic term of the scalar field, and ϕ represents an additional scalar field (k-essence model) [1]. This approach allows one to account for additional geometric effects (torsion, non-metricity) via metric-affine theories of gravity (MAG).

The main goal of the study is to integrate the extended theory of gravity with the effects of gravitational weak lensing and black hole shadow formation in a plasma medium. In order to achieve this goal, the following tasks have to be solved: Modification of the metric potential [2].

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + sin^2\theta d\varphi),$$

where the function *f*(*r*) as the form

$$f(r) = 1 - \frac{2M}{r} + \frac{\lambda}{r^2},$$

where the function $f(r)$ is of the form f and λ is the modification parameter.

Methods

In this section we studied extended gravity, namely gravity $F(R,Q)$. Earlier we have considered Lagrangians, Hamiltonians and gravitational equations for spacetime FLRU, which allows us to obtain particular and generalised cases of theories of gravitation. It is known that the classical GR, in spite of its physical and mathematical validity, faces problems (dark energy, dark matter, inflation), so its modification becomes necessary. One of variants of such modifications is the approach of metric-affine theories of gravity (MAG), in which geometry includes not only curvature, but also torsion and non-metricity. To generalise the k-essence model, we introduce a function $F(R,Q,X,\phi)$in action, where X is the kinetic term of the scalar field and ϕ is the complementary scalar field. Below is a special case allowing to describe the observational data in the neighbourhood of black holes.

This chapter provides a detailed description of a modified stationary and spherically symmetric metric used to model the gravitational field around a black hole [3]. The main attention is paid to the analytical derivation of the function $F(r)$, the analysis of the structure of horizons and the discussion of the influence of the parameter λ on the geometry of space-time. The results are supported by graphical analysis, which makes it possible to clearly see the impact of the modification on key characteristics [4].

Results and Discussion

Let us consider a stationary spherically symmetric spacetime described by the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\,(d\theta^2 + sin^2\theta d\varphi^2) \quad (1)$$

In this model, the function $f(r)$ is defined as follows:

$$F(r) = 1 - \frac{2M}{r} + \frac{\lambda}{r^2}. \quad (2)$$

Where M is mass of the black hole (taken dimensionless for convenience of calculations), λ is parameter characterising deviations from the classical Schwarzschild solution, $g(r)$ is function defined by a particular modification [5]. For example, it is possible to take $g(r) = \frac{1}{r^2}e^{-\alpha r}$, M is the mass of the black hole (we take a dimensionless value, setting $M = 1$ for convenience of calculations), λ is a parameter characterizing deviations from the classical Schwarzschild solution. For $\lambda = 0$ function (2) reduces to the standard Schwarzschild solution:

$$F(r) = 1 - \frac{2M}{r}. \quad (3)$$

The function $F(r)$ of r for various values of λ shown in Figure 1.

Analysis of the horizon structure

The horizons of a black hole are determined by the roots of the equation $F(r) = 0$. Substituting (2), we get:

$$F(r) = 1 - \frac{2M}{r} + \frac{\lambda}{r^2}. \quad (4)$$

For convenience, we multiply this equation by r^2

$$r^2 - 2Mr + \lambda = 0. \quad (5)$$

Solving the quadratic equation in r using the formula:

$$r = \frac{2M \pm \sqrt{(2M^2 - 4\lambda)}}{2} = M \pm \sqrt{M^2 - \lambda}. \quad (6)$$

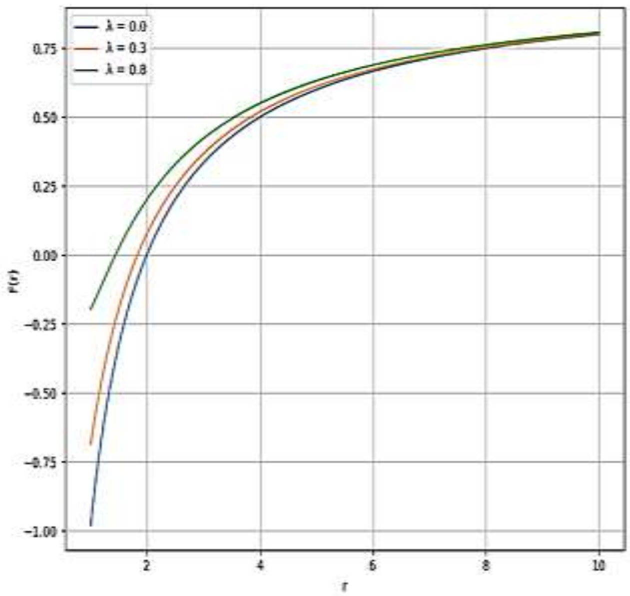

Figure 1 – The function $F(r)$ of r for various values of λ. The purpose of the graph: Visually demonstrate how changing the parameter λ affects the function $F(r)$. With an increase in λ, significant changes in the behavior of the function are observed, which affects the location of the horizons and subsequent dynamic properties [4-5].

Conditions for the existence of horizons: the roots of (6) are valid under the condition $M^2 \geq \lambda$ [6]. When $M^2 \geq \lambda$ we get two different roots: $r_+ = M + \sqrt{M^2 - \lambda}$ is the outer horizon, $r_- = M - \sqrt{M^2 - \lambda}$ is the inner horizon.

Dependence of horizon radii on λ. Schedule goal: Show how the internal and external horizons change when the parameter is varied. This allows us to evaluate how modification of the metric affects the global structure of space-time.

Figure 2 illustrates the dependence of the inner and outer horizon radii on the parameter λ.

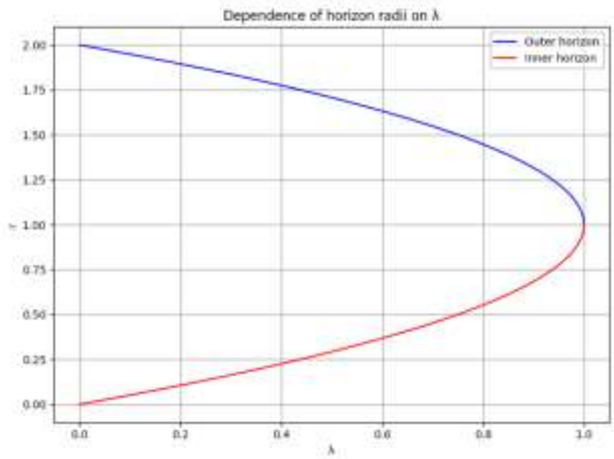

Figure 2 – The graph shows that at $\lambda = 0$, the inner horizon is zero, and the outer horizon is $2M$ (at $r = 2$). As λ increases, the distance between the horizons decreases [7].

Plasma effects and refractive index

When light propagates in a plasma medium, the refractive index $n(r)$ is determined by the following relation:

$$n^2(r) = 1 - \frac{\omega_p^2(r)}{\omega^2(r)} \tag{7}$$

where $\omega_p(r)$ is local plasma frequency, $\omega(r)$ is frequency of the photon measured by an observer located at a given point [7].

The gravitational redshift causes the local frequency of a photon to be related to its frequency at infinity ω_0 as follows:

$$\omega(r) = \omega_0\sqrt{F(r)} \tag{8}$$

where $F(r)$is defined by expression (7). Thus, the refractive index becomes a function of both the radial coordinate r and the parameter λ through $F(r)$. Figure 3 shown dependence of the refractive index $n(r)$ on r.

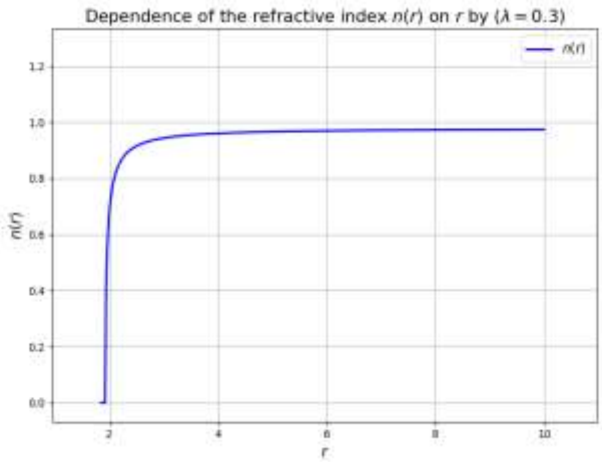

Figure 3 – This graph shows the dependence $t(k)$ from r for a fixed value of r and demonstrates how gravitational redshift and metric modification affect the refractive index where $F(r)$. Thus, the refractive index becomes a function of both the radial coordinate r and the parameter λ through $F(r)$ [7].

The photon sphere and the shadow of a black hole

A photon sphere, a region in which photons can be in closed (unstable) orbits, and calculates the size of the shadow of a black hole observed at infinity. The results obtained depend on the modified metric described by the function

$$F(r) = 1 - \frac{2M}{r} + \frac{\lambda}{r^2} \tag{9}$$

as well as from the influence of plasma effects, which are taken into account through the refractive index and plasma frequency [8].

Effective pulse parameter and photon sphere condition

In the presence of plasma, the effective pulse parameter is determined by the ratio:

$$h^2(r) = r^2\left[\frac{1}{F(r)} - \frac{\omega_p^2}{\omega_0^2}\right] \tag{10}$$

where ω_p is plasma frequency (with a homogeneous plasma it is considered a constant),ω_0 is frequency of the photon at infinity $F(r)$ is a metric function defined by formula (2). A photonic sphere is formed at the point where the $h^2(r)$ function reaches its maximum. That is, it is necessary to fulfill the condition:

$$\frac{d}{dr}h^2(r)\bigg|_{r=r_{ph}}. \tag{11}$$

Solving this equation numerically allows us to determine the radius of the photonic sphere r_{ph} [9].

Effective pulse parameter $h^2(r)$ and definition of the photonic sphere. Figure 4 shows the dependence $h^2(r)$ on the radial coordinate r or the selected value λ and the fixed ratio ω.

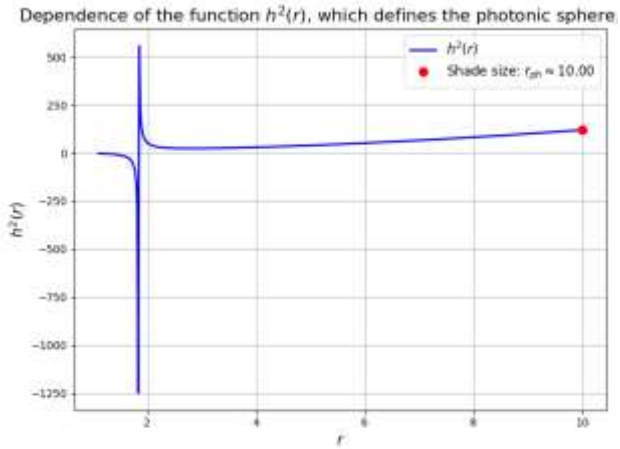

Figure 4 –The maximum of the function $h^2(r)$ corresponds to the position of the photonic sphere, which is marked on the graph [7].

Calculation of the shadow of a black hole

The size of the shadow of a black hole is determined by the angular radius that can be obtained for an observer located at infinity [10]. The angular size of the shadow α is determined by the ratio:

$$sin^2\alpha_{sh} = \frac{h^2(r_{ph})}{h^2(r_0)}. \qquad (12)$$

Where r is the distance to the observer. Provided that $r_0(F(r_0) \approx 1, \omega_p(r_0) \approx 0)$ is very large, we can assume:

$$h^2(r_0) \approx r_0^2. \qquad (13)$$

Thus, the observed shadow value is defined as:

$$R_{sh} \approx r_0 sin\alpha_{sh} = \sqrt{h^2(r_{ph})}. \qquad (14)$$

Dependence of the shadow size on the parameter λ. Graph description: For a set of values of λ, a numerical calculation of r is performed, followed by the calculation of R. The graph (Fig.5) shows how the size of the shadow changes with a variation of λ at fixed values of plasma parameters.

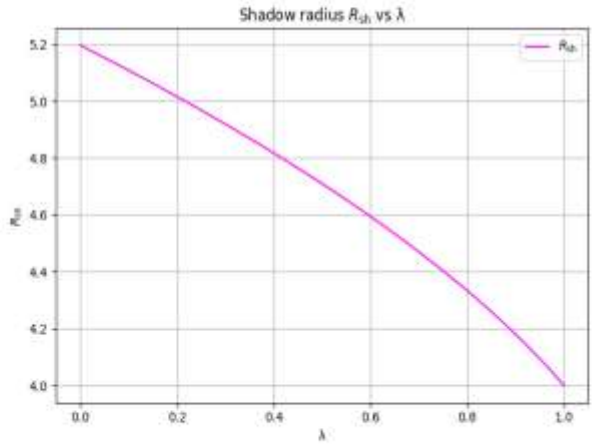

Figure 5 – The graph shows that with an increase in λ, the shadow size decreases, which is explained by a decrease in the radius of the photonic sphere. This observation is important for interpreting astronomical data and testing modified gravity models [8].

Gravitational lensing in a weak gravitational field

The process of gravitational lensing of photons in a weak gravitational field described by a modified metric.

$$F(r) = 1 - \frac{2M}{r} + \frac{\lambda}{r^2} \qquad (15)$$

Special attention is paid to the decomposition of the metric in the domain where $r \gg 2M$ and obtaining an approximate analytical expression for the angle of light deflection. The obtained results are accompanied by a graphical analysis, which makes it possible to visually assess the dependence of the angle of deviation on the pulse parameter and the influence of the parameter λ and plasma effects [11].

Decomposition of the metric in a weak gravitational field

The Minkowski metric:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \qquad (16)$$

where $n = diag(-1,1,1,1)$ is the Minkowski metric, and $h_{\alpha\beta}$ is a small disturbance.

From expression (2) we obtain an approximate representation:

$$h_{00} = \frac{2M}{r} - \frac{\lambda}{r^2},\ h_{ij} = \left(\frac{2M}{r^2} - \frac{\lambda}{r^2}\right)\delta_{ij}. \qquad (17)$$

This decomposition determines that in a weak field, the modification of the metric introduces corrections of the order of $\frac{1}{r}$ or $\frac{1}{r^2}$ to the components of the metric tensor.

Light deflection angle output

For a photon passing in a weak gravitational field, the deflection angle α can be found using an integral expression. When considering a photon moving with the momentum parameter b, the integration method along the line of rectilinear motion is often used. When plasma effects are taken into account, the expression for the deflection angle has the following form:

$$\hat{\alpha}(b) \approx \frac{4M}{b}\left(1+\frac{1}{1-\frac{\omega_p^2}{\omega_0^2}}\right)- -\frac{\pi\lambda}{4b^2}(2+\frac{1}{1-\frac{\omega_p^2}{\omega_0^2}}, \qquad (18)$$

b is momentum parameter (the distance from the center of the black hole to the line of motion of the photon, approximately $b \approx D_d\theta$, where D_d is distance to the lens, ω_0 is frequency of the photon at infinity, ω_p is plasma frequency (with homogeneous plasma it is considered a constant).

The first term in Eq.(18) corresponds to the standard deviation angle multiplied by the correction factor associated with the plasma dispersive effect. The second term is due to a modification of the metric due to the parameter λ and has the order of $\frac{1}{b^2}$.

Dependence of the deflection angle α on the impulse parameter b. The Fig.6 demonstrates that with increasing b, the angle of deviation decreases, and the influence of the additional term $\frac{\lambda}{r^2}$ leads to a noticeable change in the curve compared with the classical case (for $= 0$)

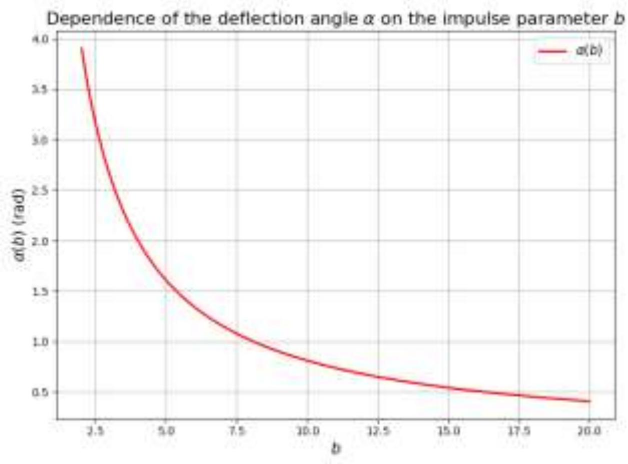

Figure 6 – This graph shows the dependence of the angle of deviation $\hat{\alpha}$ on the impulse parameter b for the selected value λ and a fixed ratio $\frac{\omega_p^2}{\omega_0^2}$.

The graph shows that as b increases, the angle of deflection α decreases. For $= 0.3$, there is a slight deviation from the classical behavior due to a modification of the metric, and the plasma coefficient increases the total angle of inclination by a factor of $\frac{1}{1-\frac{\omega_p^2}{\omega_0^2}}$ [4-8].

The lensing equation

Gravitational lensing is described by an equation relating the angular position of the source (β), the angular position of the image (θ) and the angle of deflection $(\widehat{\alpha})$. In general, the equation has the form:

$$\beta = \theta - \frac{D_{ls}}{D_S}\hat{\alpha}, \quad (19)$$

where D_{ls} is distance from the lens to the source,D_S is distance from the observer to the source.

In a weak gravitational field and provided that the impulse parameter b is approximately equal to $b \approx D_d\theta$ (where D_d is the distance to the lens), the lensing equation takes the form:

$$\beta = \theta - \frac{D_{ls}}{D_s}\hat{\alpha}(D_d\theta). \qquad (20)$$

In this expression, the angle of deviation $\hat{\alpha}$ is determined by an approximate formula:

$$\hat{\alpha}(b) \approx \frac{4M}{b}\left(1+\frac{1}{\frac{\omega_p^2}{\omega_0^2}}\right)-\frac{\pi\lambda}{4b^2}(2+\frac{1}{1-\frac{\omega_p^2}{\omega_0^2}}. \quad (21)$$

Substituting $b \approx D_d\theta$ into this expression, we obtain a nonlinear equation with respect to θ.

Numerical solution of the lensing equation

To determine the angular positions of the images θ at a given angular position of the source equation (21) is solved numerically. In practice, the equation may have several solutions corresponding to different images: primary, secondary, and (in some cases) relativistic [11-15].

Dependence of the angular positions of the images θ on the angular position of the source β is shown in the Figure 7.

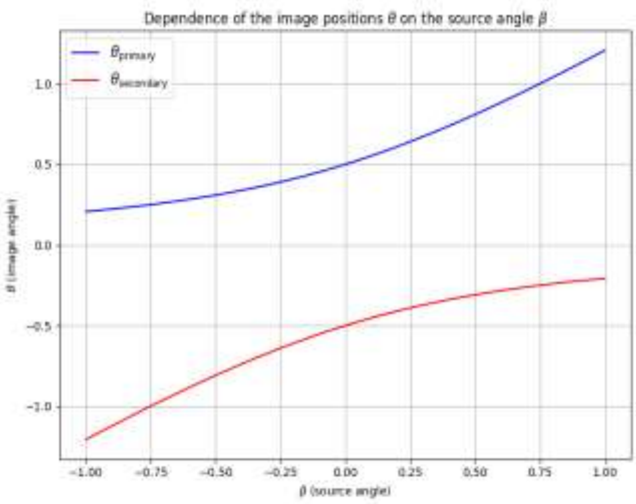

Figure 7– The graph shows the obtained numerical solutions of equation (21) for the primary and secondary images with a variation of β in a certain range. The graph allows you to see how a change in the angular position of the source leads to a shift in the positions of the images [8].

Image magnification calculation

Magnification of an image is defined as the ratio of the area of the image to the area of the source [16-18]. To a one-dimensional approximation, the magnification for each image can be written as follows [19-20]:

$$\mu = \left|\frac{\theta}{\beta}\frac{d\theta}{d\beta}\right|. \qquad (22)$$

The total magnification of the image system is determined by the sum of the magnifications of all images:

$$\mu_{tot} = \sum_i \mu_i. \qquad (23)$$

In the analytical or numerical solution of equation (22), one can calculate the dependence $\theta(\beta)$, and then, using (23), find μ.

Dependence of image magnification on the angular position of the source β is shown in the Figure 8.

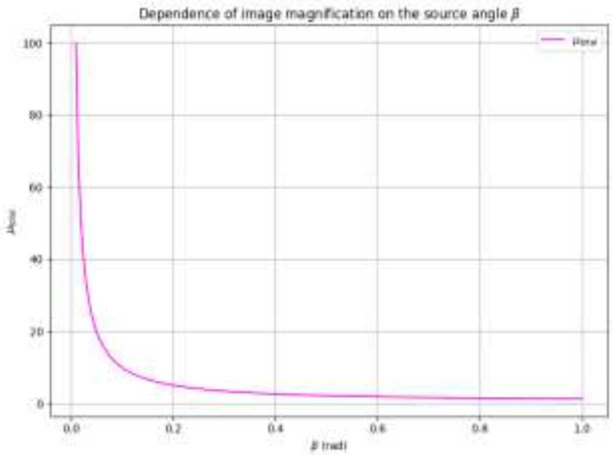

Figure 8 – This graph shows the conditional dependence of magnification for primary and secondary images depending on β. The graph allows you to visually assess how a change in the angular position of the source affects the brightness and magnification of images obtained as a result of lensing [4-8].

Conclusion

In this paper the integration of the extended theory of gravitation with the analysis of gravitational lensing and black hole shadow formation in the plasma medium was carried out. The main results of the study can be formulated as follows: Generalised theory of gravitation. The paper considers a modification of the classical general theory of relativity by introducing a function$F(R, Q, X, \phi)$ which includes not only the curvature scalar R but also the non-metricity scalar Q together with an additional scalar field ϕ (k-essence model). This approach allows us to take into account the effects of torsion and nonmetricity, which extends the scope of application of the theory in modelling gravitational phenomena under extreme conditions.

Modified metric potential. Based on the generalised action, a stationary spherically symmetric metric has been obtained in which the function $f(r)$ has the form $f(r) = 1 - \frac{2M}{r} + \lambda g(r)$.Here, the parameter λ is responsible for deviations from the classical Schwarzschild solution, and the chosen function $g(r)$ allows us to quantify the effect of the modification on the horizon structure. Numerical and graphical analyses have shown that even insignificant changes in the parameter λ can significantly change the position of inner and outer horizons, which is important for the interpretation of observational data.

Photon dynamics in the plasma medium.

The account of plasma effects is realised through the introduction of the refractive index (r), depending on the local plasma frequency and gravitational redshift: $n(r) = \sqrt{1 - \frac{\omega_p^2}{\omega^2(r)}}$, where $\omega(r) = \frac{\omega_\infty}{\sqrt{f(r)}}$.This allows to describe correctly the propagation of photons near a black hole, where the plasma effects become essential. Gravitational lensing and shadow of a black hole. The obtained results allowed us to perform a detailed analysis of the angle of light deflection, the formation of the photon sphere and the calculation of the angular size of the black hole shadow. The modification of the metric potential leads to changes in the photon dynamics, which is reflected in observational characteristics such as image shift and magnification under lensing. Practical significance and prospects.

The results of this work demonstrate that combining the extended theory of gravitation with the analysis of plasma effects and gravitational lensing allows us to obtain a more accurate description of gravitational phenomena in the vicinity of black holes. This opens new possibilities for experimental testing of theoretical models using modern astronomical observations, such as data from the Event Horizon Telescope. Prospects for further research include taking into account the inhomogeneous distribution of plasma, the influence of black hole rotation and the development of more sophisticated numerical models to analyse the strong gravitational field.

Thus, the study confirms that modification of the classical theory of gravitation by introducing additional parameters and taking into account plasma effects allow us to significantly expand the modelling capabilities of gravitational processed.

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References

1. A. Ditta, T. Xia, S. Mumtaz, F. Atamurotov, G. Mustafa, A. Abdujabbarov, Testing the Metric-Affine Gravity using particle dynamics and photon motion, Physics of the Dark Universe **41**, 101248 (2023). https://doi.org/10.1016/j.dark.2023.101248.
2. W. Liu, D. Wu, X. Fang, J. Jing, J. Wang, Kerr-MOG-(A)dS black hole and its shadow in scalar-tensor-vector gravity theory, Journal of Cosmology and Astroparticle Physics (JCAP) **08**, 035 (2024). https://doi.org/10.1088/1475-7516/2024/08/035.
3. A. Das, A. Saha, S. Gangopadhyay, Shadow of Kottler black hole in the presence of plasma for a co-moving observer, Classical and Quantum Gravity **40**, 015008 (2023). https://doi.org/10.1088/1361-6382/aca5e4.
4. O.Y. Tsupko, G.S. Bisnovatyi-Kogan, Gravitational deflection of light in a homogeneous plasma, Physical Review D **87**, 124009 (2013). https://doi.org/10.1103/PhysRevD.87.124009.
5. H. Hoshimov, O. Yunusov, F. Atamurotov, M. Jamil, A. Abdujabbarov, Weak gravitational lensing and shadow of a GUP-modified Schwarzschild black hole in the presence of plasma, arXiv preprint, arXiv:2312.10678 (2023). https://doi.org/10.48550/arXiv.2312.10678.
6. N.U. Molla, U. Debnath, Shadows and Strong Gravitational Lensing by Van der Waals Black Hole in Homogeneous Plasma, arXiv preprint, arXiv:2212.02104 (2022). https://doi.org/10.48550/arXiv.2212.02104.
7. Q.-M. Fu, S.-W. Wei, L. Zhao, Y.-X. Liu, X. Zhang, Shadow and weak deflection angle of a black hole in nonlocal gravity, Universe **8**, 341 (2022). https://doi.org/10.3390/universe8070341.
8. X.-M. Kuang, Z.-Y. Tang, B. Wang, A. Wang, Constraining a modified gravity theory in strong gravitational lensing and black hole shadow observations, arXiv preprint, arXiv:2206.05878 (2022). https://doi.org/10.48550/arXiv.2206.05878.
9. A. Kumar, A review on analytical studies in gravitational lensing, Frontiers in Physics **11**, 1113909 (2023). https://doi.org/10.3389/fphy.2023.1113909.
10. Y. Zhang, L. Chen, M. Li, Gravitational Lensing around a Dual-Charged Stringy Black Hole in Plasma Background, European Physical Journal C **85**, 4, 101 (2025). https://doi.org/10.1140/epjc/s10052-025-13362-9.
11. S. Kumar, R. Patel, M. Sharma, Shadow and weak gravitational lensing of a rotating regular black hole in a non-minimally coupled Einstein–Yang–Mills theory in the presence of plasma, Classical and Quantum Gravity **39**, 045001 (2025). https://doi.org/10.1088/1361-6382/abca12.
12. S.L. Rubinshtejn, Osnovy obshhej psihologii [Basics of general psychology] (SPb: Piter.Kom., 1999), 720 p. (in Russ).
13. A. Einstein, The foundation of the general theory of relativity, Annalen der Physik **49**(7), 769–822 (1916). https://doi.org/10.1002/andp.19163480702.
14. S. Weinberg, Cosmology, (Oxford, Oxford University Press, 2008).
15. C.W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation (San Francisco: W. H. Freeman and Co., 1973).
16. S. Carroll, Spacetime and geometry: An introduction to general relativity (San Francisco: Addison Wesley, 2003).
17. K.K. Yerzhanov, B. Meirbekov, G. Baurzhan, R. Myrzakulov, Cosmological solutions of F(R,T) gravity model with k-essence, Journal of Physics: Conference Series (2019). https://doi.org/10.1088/1742-6596/1391/1/012070.

18. K.K. Yerzhanov, G. Baurzhan, A. Altaybaeva, R. Myrzakulov Inflation from the symmetry of the generalized cosmological model, Symmetry **13**(8), 1433 (2021). https://doi.org/10.3390/sym13081433.

19. D. Momeni, K.K. Yerzhanov, R. Myrzakulov, Quantized black hole and Heun function, Canadian Journal of Physics **90**(9), 877–881 (2012). https://doi.org/10.1139/p2012-069.

20. T. Ibrokhimov, Z. Turakhanov, F. Atamuratov, A. Abdujabbarov, K.K. Yerzhanov, G. Baurzhan, A. Abduvokhidov, Testing regular scale-dependent black hole space time using particle dynamics: Shadow and gravitational weak lensing, Physics of the Dark Universe **47**, 101778 (2025). https://doi.org/10.1016/j.dark.2024.101778.

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