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Fluctuations of electron energy levels in the interger quantum hall effect

The level statistics in the regime of the quantum Hall effect is studied. The critical exponent of the localization length is found by analyzing the lowest Landau band. By scaling procedure for different system sizes we find the spectral compressability at the plateau-plateau transition. It turned out to be scale-invariant. Obtained results are generalized for other special dimensionalities. Our findings are characteristic of the critical unitary class of universality. For two-dimensional systems the tail of the level spacing distribution resembles the Poisson distribution. It is similar to that of three-dimensional systems, although the exponential rate is as twice as large. Fluctuations of the energy levels are distinct from the classical Gaussian unitary ensemble data and reflect the multifractal nature of the electron wave functions.

Key words: electron conductivity, critical phenomena, two-dimensional electron gas, quantum Hall effect, critical index, energy level statistics, unitary ensemble.

Introduction

It is known that the fundamental symmetries under the time reversal operations pertain in disordered systems, while spatial geometrical symmetries, which can exist in crystalline systems, are completely lost. These level statistics are studied in the frame of the random matrix theory [1], where the energy levels are represented by the eigenvalues of model Hamiltonian matrices whose elements are randomly distributed. Within the random matrix theory the level statistics have universal properties depending only on the fundamental symmetry of the system. If the system is invariant with respect to time reversion, there are two universality classes - orthogonal and symplectic ones. The former is realized, when there is no spin-orbit interaction (i.e. the spin of the particle is integer), the latter applies, when the spin of the particle is half odd integer and there exists a spin-orbit interaction. When the time reversal symmetry is broken (e.g. by the presence of magnetic impurities or an external magnetic field) the corresponding universality class becomes of unitary type.

It is usually believed that the level statistics of extended states are well described by the random matrix theory on the basis of the Gaussian ensembles [2]. The strong level repulsion is characteristic of the level statistics in the delocalized (or metallic) regime. This can be expressed by the linear behavior of the level number variance $\langle \delta N_2(E)\rangle = k\langle N(E)\rangle$ and by the power-law behavior s^{β} of the level spacing distribution function $P(s)$ in the small-s spacing region.

It is well-known that the level repulsion parameter β takes on the values 1, 2 and 4 for the Gaussian orthogonal ensemble (GOE), the Gaussian unitary ensemble (GUE) and the Gaussian symplectic ensemble (GSE), respectively [3]. This fact represents that the level correlation in the delocalized regime is quite strong. That means, the probability to find two levels in an infinitesimally small distance from each other is vanishing, i.e. $P(s) = 0$.

On the other hand, the electron energy levels in the localized regime are completely uncorrelated. This is due to negligible spatial overlapping between wave functions of corresponding localized electron states. In this case the level statistics become Poisson-like and the level number variance is exactly equal to the average level number within the fixed energy interval $\langle \delta N_2(E)\rangle = \langle N(E)\rangle$ [4]. From these two limiting behaviors of the level statistics it is clear that the properties of the localization-delocalization transition in disordered systems can be analyzed in terms of the level statistics.

Shklovskii and co-workers [5] have analyzed the level spacing distribution in three-dimensional Anderson model by changing the system size and the strength of disorder. They have found that the level spacing distribution function satisfies a certain scaling property. From the scaling properties one can extract the critical exponent of the localization length at the metal-insulator transition. Their computed value of the critical exponent is consistent with other investigations [6,7]. In this paper the level statistics in the regime of the quantum Hall Effect (QHE) is studied, in terms of the level number variance and the level spacing distribution.

Critical level number variance in QHE

Before starting to analyze the statistical properties of the spectra close to the center of the Landau bands, one should construct the form of the density of states with the goal to find the critical energies. It is known that points of the plateau-plateau transitions for the transverse conductivity σ_{xy} in the integer quantum Hall Effect coincide with zeros of the longitudinal conductivity σ_{xx} [8]. This is a prerequisite of the maxima for the corresponding values of the density of electron states $\rho(E)$. In fact, in the theoretical lattice representation the energies where the density of states has extremities do not necessarily fall on the critical points of the QHE-insulator transition. This is because the density of states, as a first mathematical moment, is not sensitive to the critical features of the transition.

This is the higher order cumulants (e.g. variance as a second moment of the density of states,) that are responsible for signalizing the criticality. The values of energies in the electron spectrum where the transition occurs are called the critical energies E_c. At these critical energies the statistical properties of the electron spectrum are such that the variance of the number of energy levels in a given energy interval E exhibit finite-size scaling behavior.

Exactly at the critical energy E_c where the QHE-to-insulator transition occurs in the lowest subband I have found the linear behavior of the level number variance:

$$\langle \delta N_2(E) \rangle = k_{QHE} \langle N(E) \rangle , \quad \text{with} \quad k_{QHE} = 0.13 \pm 0.01. \tag{1}$$

The results of the extensive numerical simulations are shown on Figure 1. Different sizes of the two-dimensional system have been modeled in order to confirm the scaling ideas ranging from L=50 up to L=400.

On the one hand, this proportionality law Eq. (1) is quite similar to the critical three-dimensional case with $k \approx 0.27$. On the other hand, the linearity resembles the insulating limit with $k \equiv 1$, i.e. Poisson law of uncorrelated variables. All these cases are depicted in the Figure 1. It is seen that in the log-log scale the asymptotic energy-large behavior has the slope equaling unity. Other limiting cases of the GUE and the Poisson are shown as well. Since all points lie on the same curve, we can conclude that the spectral fluctuations in the critical region do not depend on the size of the system. This scale-invariance is signature of the universality of the critical unitary statistics.

For the short range correlations the number variance follows the random matrix approach, however with another prefactor distinct from the Wigner formula for $P_{\mathrm{GUE}}(s)$ [9]. This can be seen in the inset of Figure 1. The linear slope corresponds to $\langle \delta N_2(E) \rangle = k_{QHE} \langle N(E) \rangle$ with $k \approx 0.15$. This value is pretty close to the one obtained earlier according to Eq. (1). The crossover from the GUE limit valid for short-range correlations to the critical one corresponding to the long range correlations does not depend on the size of the system and lies in the range of the mean level number between $\langle N(E) \rangle \approx 1$ and $\langle N(E) \rangle \approx 10$. The weak localization corrections to the Gaussian Unitary Ensemble are evaluated in the work [10].

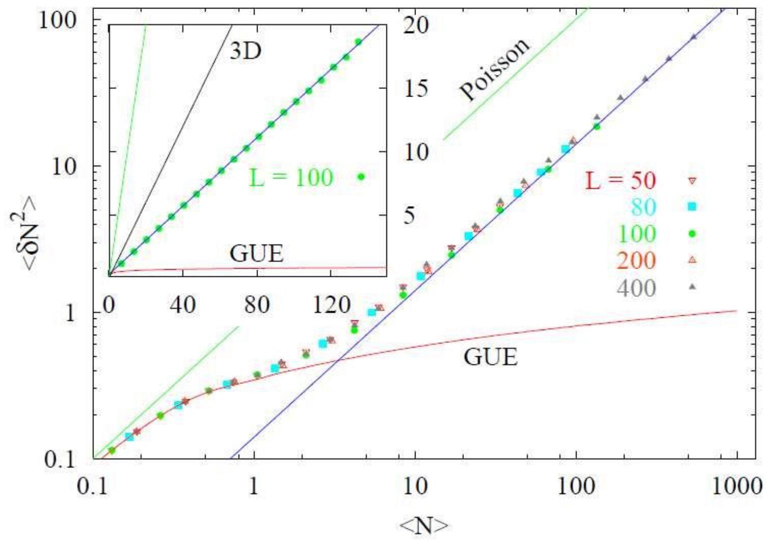

Fig.1. Level-number variance ⟨$\delta N_2(E)$⟩ as a function of the average number of electron levels ⟨$N(E)$⟩ in an energy interval of a given width E at the critical critical energy E_c=-3.455 of a two-dimensional system under strong magnetic field of the magnitude α=0.1 for different linear size L, shown by various colors (the unitary case β=2). The error-bars equal the sizes of the symbols. Violet and green lines: the GUE result and the Poisson law, respectively. Blue straight line: asymptotical critical number variance. *Inset*: the same dependence in linear scale for the system size L=100. Dark-blue straight line corresponds to the critical level number variance for the three-dimensional case

The values for the critical number, by other words the variance-over-mean which is equal to the critical prefactor k are very important entities for the scaling hypothesis and for the computation of the critical parameters like the multifractality dimensions and the critical exponent of the localization length [11] are collected in the tables of my papers [9,12] together with other cases of the Anderson-type transitions. It is clear that the linear law for level number variance is tightly connected to the two-point correlation function, which has been computed elsewhere [13].

Relation to the multifractality of the electron wave functions

In the paper of Chalker et al [14] it has been proposed that the properties of the wave functions are related to the level statistics. Especially for anomalous diffusion found in the critical this relation connects the moments of the amplitude of the electron eigenstates to the level number variance in the following way:

$$k = (d-D_2)\,/2d, \qquad (3)$$

where d is the spatial dimensionality and D_2 is the second moment of the spectrum of multifractality (the fractal dimension of the second order).

For the quantum Hall systems these quantities have intensely studied and are well known. By a precise computation of the level statistics one can test the proposed theory. Unfortunately, at present no reliable analytical theory exists for the tail of the level spacing distribution that will be discussed in the next section. The interesting value is the first derivative of the level number variance with respect to energy. Since we believe in the above-mentioned linearity, for simplicity, we choose the ratio ⟨$\delta N_2(E)$⟩/⟨$N(E)$⟩ “variance-over-mean”. It should saturate to the derivative in the limit of large <N> and provide us with the precise value of the prefactor k_{QHE}.

To estimate the asymptotic, on the one hand, it is required to use an energy interval that comprises as many eigenvalues as possible. On the other hand, the critical region should be kept as narrow as possible to avoid undesirable mixing of the localized states. For that, of course, one needs to diagonalize Hamiltonian matrices of huge size. Therefore we computed a small part of the spectrum of the two-dimensional square lattice of linear size L=400. By trying to increase the number of the eigenvalues, one is tempted to broaden the width of the energy interval around the critical energy E_c=-3.455. The latter procedure unavoidable leads to the accuracy looses (error-bars grows) for the critical statistics because then the energy dependence of the level statistics comes into play.

We have found reasonable values for the energy width and demonstrate the results in the Figure 1. The numerical results for the level number variance $\langle \delta N_2(E)\rangle$ suffers strongly from the fact that one can use only a finite number of energy levels, up to $\langle N\rangle$=400-500 as a maximum. This causes auxiliary correlations that could affect and falsify the final result for large energies [15]. One can extend the range of N which produces reliable results by studying the quantity $\langle \delta N_2(N^*)\rangle$, where

$$N^* \equiv \langle N\rangle(1-\langle N\rangle/N_0)$$

is the reduced mean number of the electron levels with N_0 being the total number of levels in the system.

The energy interval which we determined to be the largest one still showing the critical fluctuations contains approximately 440 levels. Due to the difficulties discussed above one expects only the data up to N=140 to be reliable. However at this value the variance-over-mean does not saturate to its limiting value. So the best one can provide is an upper bound for the prefactor, which is roughly 0.13. But this value is not sufficient to distinguish between two predictions k =0.15 (of above made estimations) and k= $(d-D_2)/2d$=0.09 [16]. To improve the accuracy and to proceed further one needs a number of level at least more than 1000-3000 levels only in the critical region, which would correspond to a unprecedented scale – to a matrix size of about 2000x2000 [17], which is at present not achievable with nowadays computer facilities and power.

The level spacing distribution in the quantum Hall effect

In contrast to the level number variance, which can be expressed as an integral of the two-level correlation function, the level spacing distribution $P(s)$ possess all of the orders of the correlations including both the two-point correlations and the higher terms. Therefore the long-range behaviour should be different from the one for the level number variance. The long range correlations correspond to the asymptotic limit of large energies. In the case of the distribution of the distances between the nearest-neighbouring energy levels, i.e. $P(s)$ we deal with large range of spacings s. That is why it is imperative to carry out the computer simulations for $P(s)$ for the asymptotically s-large region, which is quite challengeable task [18]. In the Figure 2 we show the results of our calculations for the critical level spacing distribution in the regime of the Quantum Hall effect.

One can observe the linear behaviour for the logarithm of $P_{QHE}(s)$, which is quite similar to the linear slope for 3D Anderson-type metal-insulator transition (shown by the blue line). All computed values (various symbols in the Figure 2) are lying on the same curve that means a size-independence of the level spacing distribution. In the statistical-physical sense this is a demonstration of the criticality.

As for the other critical ensembles [9,19] we have to perform a fitting procedure for the tail of $P(s)$. For the Quantum Hall Effect-to-insulator transition we have the following law:

$$P_{QHE}(s) = \exp(-\gamma s), \qquad \text{with} \quad \gamma \approx 4.2 \tag{2}$$

Apparently this law deviates from the Wigner surmise (shown by red line in the Figure 2). However for small spacings s (or energies) it shows a quadratic behavior as expected for unitary symmetry.

For large spacings *s* the value of α is quite close to the one found for two-dimensional systems with symplectic symmetry (i.e. with the strong spin-orbit coupling) [20].

In investigating the three-dimensional Anderson model of localization [21] it has been suggested in our previous paper s[15,22] that the exponential decay rate γ is related to the variance-over-mean coefficient via the formula $\gamma=1/2k$. Comparing the equation (1) and (2) one can confirm this relation also for the Quantum Hall effect, although the discrepancy a bit larger.

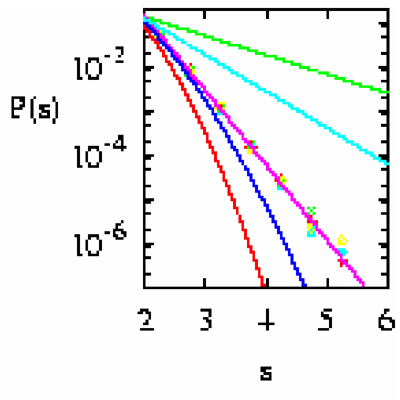

Fig.2. Distribution function of the inter-level distances *P*(*s*) in the vicinity of the critical energy Quantum Hall-to-insulator transition in the lowest Landau band for various sizes of the two-dimensional system *L*=50, 80, 100, 200 (shown by symbols of different type). Only asymptotic region of *P*(*s*) for large *s* is plotted. The Poisson law is depicted by green straight line. The Wigner surmise for GUE is depicted by red. Blue line corresponds to the asymptotic behavior of the critical orthogonal *P*(*s*) for a three-dimensional system

Summary

In conclusion, we were able to show the existence of a critical level statistics in the quantum Hall system, at least for the lowest Landau band. The spectral fluctuations in the critical region do not depend on the size of the system and therefore universal. They are distinct from the classical canonical GUE data and reflect the multifractal nature of the eigenstates. In the future we plan to extract the critical exponent of the localization length in the QHE regime using the scaling finite-size properties of the level statistics. The level spacing distribution has the simple exponential sub-poissonian decay analogous to the typical Anderson-transition problems. However it should be noticed here that, although substantial progress in the understanding of critical properties in the transition from insulator to the quantum Hall effect has been achieved, nevertheless many issues of the Anderson-type of the metal-insulator transition [23-26] are still considered as being open and unsolved.

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И.Х. Жәрекешев

Бүтінсандық кванттық Холл эффектісіндегі электрондық деңгейлердің флуктуациясы

Холлдың кванттық эффектісіндегі энергия деңгейлерінің статистикасы қарастырылған. Ландаудың төменгі зонасының анализі бойынша локализация ұзындығының критикалық экспоненті табылды. Скейлинг әдісімен өткелдегі көлденең кедегісінің үстірт аралық спектралдық корреляциялау функциясын зерттелген. Алынған нәтижелер басқада өлшем бірліктерімен жинақталған. Оның кең көлемді-инвариантты екені белгілі болды. Қол жеткізілген нәтижелер критикалық унитарлық класс жанжақтылығы үшін өзіне тән ерекшеліктермен сипатталады. Екіөлшемдік жүйелер үшін асимптотикалық кему Пуассонның үлестірілуіне ұқсайды. Ол, экспоненциалдық көрсеткіші екі есе жоғары болса да, үшөлшемді жүйелердің кемуіне ұқсас келеді. Энергия деңгейдерінің флуктуациялары классикалық Гауссовтік унитарлық ансамблі мәліметтерінен ерекшеленеді. Олар электронның толқынды функциясының мультифрактaлдық табиғатын әсерлейді.

Түйін сөздер: электрондық өткізгіштік, критикалық құбылыстар, екі өлшемдегі электрондық газ, Холлдың кванттық эффектісі, критикалық индекс, денгейлер статистикасы, унитарлық ансамбл.

И.Х. Жарекешев

Флуктуации электронных уровней энергии в целочисленном квантовом эффекте Холла

Изучается статистика уровней энергии в квантовом эффекте Холла. По анализу нижней зоны Ландау найдена критическая экспонента длины локализации. Методом скейлинга конечного размера мы исследуем спектральную сжимаемость на переходе между различными плато поперечного сопротивления. Оказалось, что она является масштабно-инвариантной. Полученные данные обобщены на другие пространственные размерности. Наши результаты характерны для критического унитарного класса универсальности. Для двумерных систем асимптотический спад похож на распределение Пуассона. Он аналогичен спаду трехмерных систем, хотя экспоненциальный показатель в два раза больше. Флуктуации уровней энергии отличаются от данных для классического Гауссового унитарного ансамбля. Они отражают мульти фрактальную природу волновой функции электрона.

Ключевые слова: электронная проводимость, критические явления, двумерный электронный газ, квантовый эффект Холла, критический индекс, статистика уровней, унитарный ансамбль.