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Qualitative analysis of the eigenvalue problem for two coupled Ginzburg-Landau equations

Eigenvalue problem for two coupled Ginzburg-Landau equations is numerically investigated. The fixed points of corresponding equations system are found. The classification of these points is made. It is shown that the investigated equations set has local minima, global minima, local maximum, and the points refer to saddle points. The phase portraits of corresponding ordinary differential equations and the dependence of some parameters of the equations system and the total energy on the initial values are given.The profiles of dimensionless energy density of the equations set for the different initial values of are given. The dependence of the parameters of the system   and the total energy   on the initial values of   is investigated.

Key words: Ginzburg-Landau equations, eigenvalue problem, qualitative analysis

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Гинзбург-Ландаудың байланысқан екі теңдеулері үшін меншікті мәндері есебінің сапалық талдауы

Гинзбург-Ландаудың өзара әсерлесетін екі теңдеулері үшін меншікті мәндері есебі сандық зерттелді. Теңдеулер жүйесіне сәйкес сындық нүктелер табылды. Осы нүктелердің классификациясы жасалынды. Зерттелген теңдеулер жүйесінде локалды және глобалдыминимумдар мен максимумдар, сонымен қатар ер пішінді орналасқан нүктелері бар болатындығы көрсетілді. Осы қарапайым дифференциалдық теңдеулерге сәйкес фазалық портреттер алынды, сонымен қатар толық энергияның тәуелділігі және теңдеулер жүйесінің кейбір параметрлерінің бастапқы мәндерге тәуелділігі алынды. Әртүрлі бастапқы мәндер үшін скалярлық өрістерде қарастырылатын энергияның өлшемсіз тығыздығының профилі келтірілген. Жүйенің   параметрлерінің және   толық энергияның бастапқы мәндерге тәуелділігі зерттеледі.

Түйін сөздер: Гинзбург-Ландаутеңдеулері, спектралдық есеп, сапалық талдау.

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Качественный анализ задачи на собственные значения для двух связанных уравнений Гинзбурга-Ландау

Численно исследована задача на собственные значения для двух взаимодействующих уравнений Гинзбурга-Ландау. Найдены критические точки соответствующей системы уравнений. Произведена классификация этих точек. Показано, что исследованная система уравнений имеет локальные и глобальные минимумы и максимумы, а также седловые точки. Получены фазовые портреты, соответствующие этим обыкновенным дифференциальным уравнениям, а также получена зависимость полной энергии и некоторых параметров системы уравнений от начальных значений. Приведены профили безразмерной плотности энергии рассматриваемых скалярных полей для различных начальных значений. Исследуется зависимость параметров системы   и полной энергии   от начальных значений   .

Ключевые слова: уравнения Гинзбурга-Ландау, спектральная задача, качественный анализ

UDC 519-7

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QUALITATIVE ANALYSIS OF THE EIGENVALUE PROBLEM FOR TWO COUPLED GINZBURG-LANDAU EQUATIONS

Introduction

Systems with coupled scalar fields hold their certain interest in physical applications. In particular, such systems are used in constructing particle models and their interactions within the framework of quantum field theory [1]. In different aspects such systems have been already repeatedly considered, see for example Refs. [2-8]. Here we present a qualitative study of one of such systems having the potential energy in the form:

$$\rule{1cm}{0.4pt} \qquad \rule{1cm}{0.4pt}$$

$$\rule{1cm}{0.4pt} \tag{1}$$

where φ, ψ are two real scalar fields (usual or phantom/ghost ones), m are some constants. This potential had been used earlier by us in treatments of models of cosmological and astrophysical objects in general relativity. These researches showed that: (a) for the four-dimensional case there exist regular spherically and cylindrically symmetric solutions [9-11], and also cosmological solutions [12,13] both for usual and phantom scalar fields; (b) for the higher dimensional cases there exist the thick brane solutions [14-17] supported by usual and phantom scalar fields.

From the physical point of view, these solutions exist because of the special form of the interaction potential (1) having two local and two global minima. It means that there are two different vacua. At the infinity, as the radial coordinate $x \to \pm\infty$, these scalar fields are located in that vacuum in which their are in the local minimum. The existence of regular solutions with finite energy is only possible for certain (eigen) values of parameters of a problem. Such eigenvalue problems were solved by us using the shooting method (see for details of the shooting procedure in Ref. [11]).

Note that the type (1) potential has been also used in the paper [18] for modeling superconductivity using two coupled Ginzburg-Landau equations. When the interaction between the fields in (1) is excluded, i.e. at $\Lambda_3 = 0$, one has two uncoupled type Ginzburg-Landau equations. On the other hand with an account taken of the interaction there are new effects, in particular, the presence of regular solutions, not present in the case of one Ginzburg-Landau equation.

Use of spherical and cylindrical symmetries in the above papers leads to obtaining the sets of coupled non-autonomous ordinary differential equations whose qualitative study is quite complicated. Here we consider a simplified problem when the system with the potential (1) is examined in cartesian coordinates. This allows to get an autonomous system of two second order ordinary differential equations for which it is possible to perform the qualitative analysis and estimate the general behavior of the system. The goal of the paper is the equalitative investigation of autonomous ordinary differential equations describing plane solutions for two coupled Ginzburg-Landau equations.

Qualitative analysis

Let us consider the physical system with the potential (1) whose Lagrangian can be presented in the form

$$\underline{\qquad} \qquad \underline{\qquad} \tag{2}$$

where $\varepsilon_1, \varepsilon_2 = \pm 1$, and the plus sign refers to usual scalar fields, and the minus sign - to phantom/ghost ones. The energy-momentum tensor of the system is:

$$\tag{3}$$

The corresponding field equations in cartesian coordinates can be written as:

$$\underline{\quad}\underline{\quad} \tag{4}$$

$$\underline{\quad}\underline{\quad} \tag{5}$$

Choosing the initial value of the scalar field φ and introducing the dimen-sionless variables ϕ

$\underline{\quad}$
$,\qquad /\varphi$, and also
$,$ let us rewrite the system in the form (taking into account the expression for the potential from (1)):

$$\tag{6}$$

$$\tag{7}$$

$$\underline{\quad} \tag{8}$$

$$\underline{\quad} \tag{9}$$

The fixed points of this system are:

$$\tag{10}$$

$$\tag{11}$$

$$\tag{12}$$

$$\tag{13}$$

$$\tag{14}$$

$$\underline{\qquad\qquad}$$

$$\underline{\qquad\qquad}$$

$$\underline{\qquad\qquad}$$

$$\underline{\qquad\qquad} \tag{15}$$

The points A, B refer to local minima, the points C, D are the global minima, E is the local maximum, and the points F refer to saddle points. Let us designate the values of the potential (1) at these points as V_i where the index corresponds to letters from A to F, and and .
Obviously that

$$\tag{16}$$

It is also possible to find the following relations:

$$\underline{\qquad\qquad}$$

$$\underline{\qquad\qquad}$$

$$\underline{\qquad\qquad\qquad}$$

Dividing the third expression by the first one and the second one and taking into account the condition (16), one can show that the conditions guaranteeing the existence of the local and global minima are: (i) for the local minimum: λ ; (ii) for the global minimum: .

Next, to determine the type of the fixed points it is necessary to make a characteristic matrix of the system (6)-(9). Using this matrix, one can write out the characteristic forth order

algebraic equation. Then the values of roots of this equation k_j determine the type of the fixed points. In our case we have the following roots:

(i) at the points A, B (the local minimum):

(iv) at the points F (the saddle points):

(ii) at the points   (the global minimum):

(iii) at the point E (the local maximum):

Taking into account the above mentioned conditions of existence of the local and global minima, one can make the classification of the fixed points presented in table I.

Table 1 – The classification of the fixed points of the system (6)-(9) for different values of the parameters ε_1 and ε_2.

	Points A, B	Points C, D	Point E	Point F
	Unstable node	Unstable node	Saddle	Depends on the values of the parameters
	Saddle	Saddle	Saddle	Depends on the values of the parameters
	Saddle	Saddle	Saddle	Depends on the values of the parameters
	Saddle	Saddle	Unstable node	Depends on the values of the parameters

Numerical analysis

From the point of view of obtaining a set of solutions (but not only of two integral curves as in a case of saddle fixed points) type "unstable node" fixed points are more interesting. In the model under consideration, they are situated: (i) at the points of the local (A, B) and global (C, D) minima at positive $\varepsilon_1, \varepsilon_2$, i.e. for usual scalar fields; (ii) at the point of the local maximum E at negative $\varepsilon_1, \varepsilon_2$, i.e. for the case of phantom/ghost fields. Note that in the latter case negative ε , ε effectively correspond to the system with usual fields but with a reversed sign of the potential (1). In this case the point of the local maximum E becomes the point of the local minimum, and solutions asymptotically tend to that point.

Table 2 – The initial values of χ_0 and the corresponding values of the parameters   for the system (6)-(9).

1	1.0	0.3	1.25104535	1.1056305	0.0441854
2	1.0	—	1.4544857	1.1878968	0.148259
3	1.0	—	1.736266	1.30665	0.41906
4	1.0	—	1.9628773	1.40650056	0.76536
5	1.0	—	2.158048	1.495301394	1.17074
6	1.0	1.0	2.33213652	1.5764432135	1.62578
7	1.0	—	2.4908109	1.6518053896	2.1246
8	1.0	—	2.63757479	1.7225756427	2.66294

All the fixed points, being the stationary points of the system (6)-(9), are situated at $\pm\infty$. Then static solutions, if they exist, should start from these points. One of physically interesting problems is a Z_2 symmetric solution. In this case the symmetry plane is chosen at $= 0$ where the derivatives $\phi'(0) = \chi'(0) = 0$. Examples of such solutions are presented in Fig. 1. Using the energy-momentum tensor (3) and the above dimensionless variables, the dimensionless energy density can be derived in the following form

$$\overline{\quad} \quad \overline{\quad} \quad \overline{\quad} \quad \overline{\quad}$$

$$\overline{\quad} \qquad \overline{\quad} \quad \overline{\quad} \qquad (17)$$

where the constant const from the potential (1) is chosen equal to $-(\lambda_2/4)\mu_2^4$ to make the energy density to be equal to zero at infinity. Using this expression, one can plot the corresponding graphs for the energy presented in Fig. 2. The total energy (mass) of the system is defined by the expression

Calculating this integral, the values of the total energy presented in the last column of table II have been found. Also, using the table, one can plot the graphs of dependence of the parameters $,\mu$ and the total energy M on the initial values of χ_0 presented in Fig. 3. The corresponding phase portraits are shown in Figs. 4, 5.
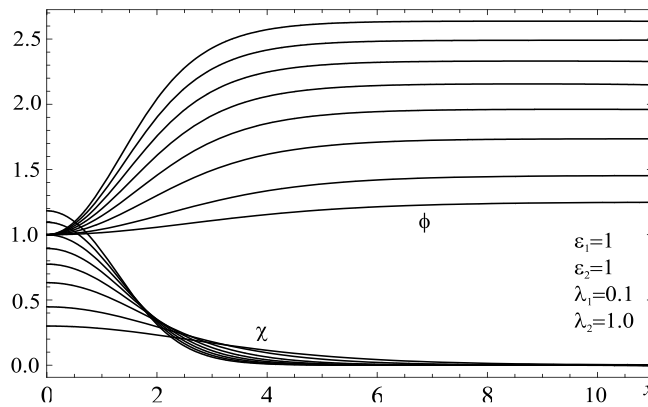

Fig. 1 − The scalar fields ϕ, χ from the system (6)-(9) for the different initial values of χ_0 taken from table II. Asymptotically, as $x \to \pm\infty$, the scalar field $\chi \to 0$, and ϕ goes to values of μ from table II corresponding to the local minimum A from (10).
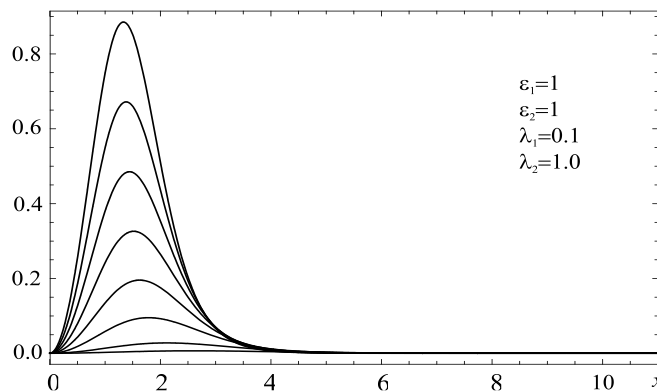

Fig. 2 − The dimensionless energy density ε of the system (6)-(9) from (17) for the different initial values of χ_0 taken from table II. The top line corresponds to the greatest $\chi_0 = \sqrt{1.4}$, and the bottom one - to the lowest $\chi_0 = 0.3$.
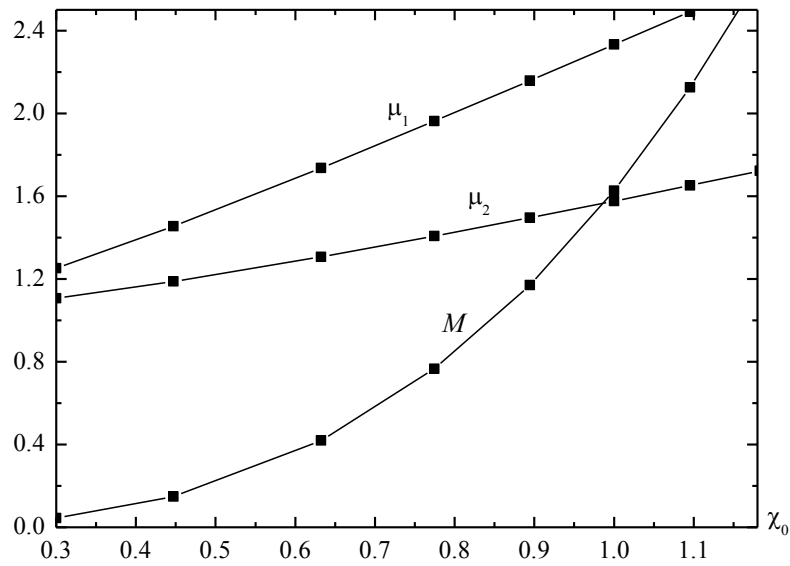
Fig. 3 – The dependence of the parameters of the system   and the total energy M on the initial values of χ_0 for the system (6)-(9). The data are taken from table II.
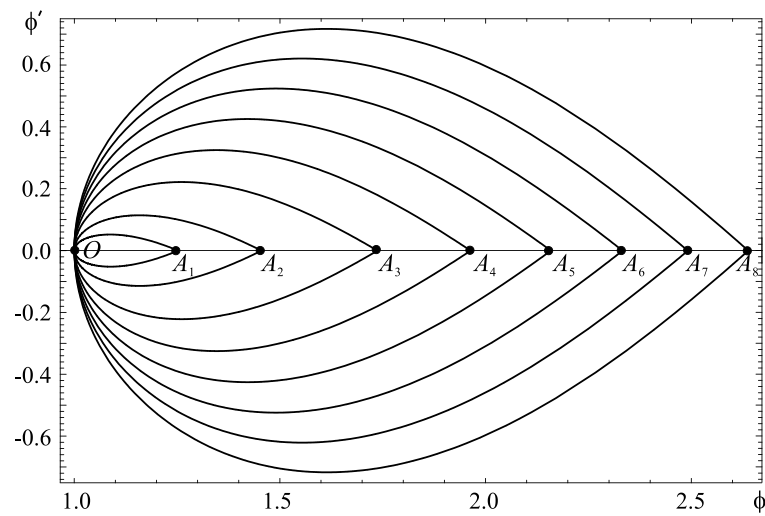

Fig. 4 – The phase portrait of the system 6)-(9) for the scalar field ϕ plotted using the parameters from table II. The points A_i correspond to the fixed point A from (10). All solutions go to these points at $x \to \pm\infty$ starting with the different initial values at the point O corresponding to the origin of coordinates $x = 0$. The index i runs over 8 in accordance with numeration from table II.
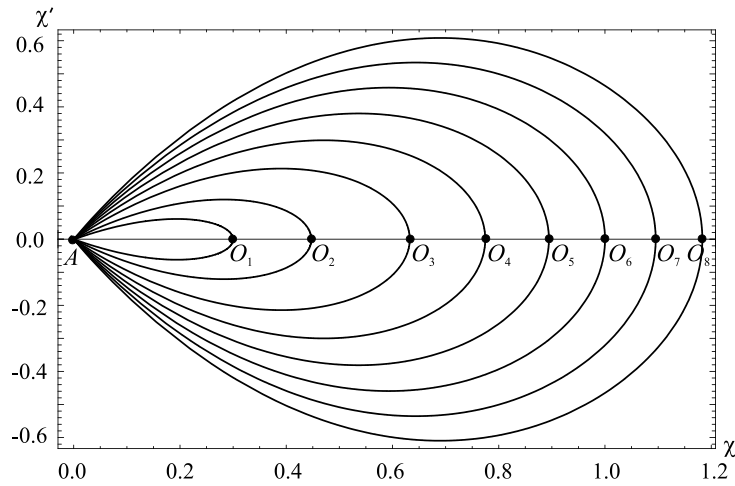
Fig. 5 – The phase portrait of the system 6)-(9) for the scalar field χ plotted using the parameters from table II. The points O_i correspond to the origin of coordinates $x =$ with the different initial values taken from table II. Asymptotically, as $\pm\infty$, the solutions tend to the fixed point from (10) where 0. The index i runs over $i = 1..8$ in accordance with numeration from table II.

The system (4)-(5) allows introducing another dimensionless variables. Namely, introducing the dimensionless variables $\bar{\phi}$

and also λ , one can rewrite this system as follows:

$$(18)$$

$$(19)$$

$$\qquad (20)$$

$$(21)$$

The values of the parameter μ and initial conditions $\bar{\phi}(0)$ and $\bar{\chi}(0)$ at which regular solutions do exist can be obtained from the values presented in table II by corresponding rescaling the variables: $\bar{\phi}$

$/\mu$. The corresponding new values of the mentioned parameters are presented in table III. Using this table, one can plot the graphs of dependence of the initial values on presented in Fig. 6.

Table 3 – The initial values and corresponding to them values of the parameter μ for the system (18)-(21).

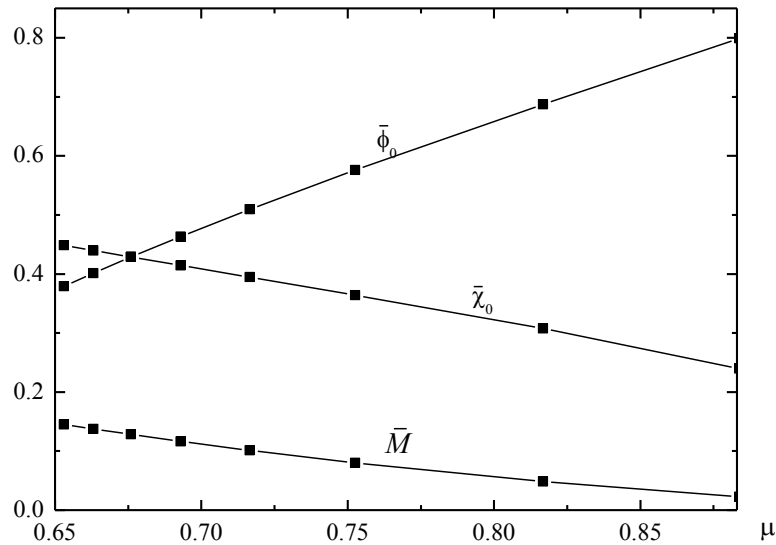|			
0.799332	0.239799	0.883765	0.0225663
0.687528	0.307472	0.816713	0.0481829
0.575949	0.364262	0.752563	0.0800622
0.509456	0.394623	0.71655	0.101201
0.463382	0.414461	0.692895	0.116487
0.428791	0.428791	0.675965	0.128174
0.401476	0.439795	0.66316	0.137485
0.379136	0.4486	0.653091	0.145127

Fig. 6 – The dependence of the initial values and the total energy $\bar{M}$ on the value of the parameter μ for the system (18)-(21). The data are taken from table III

Summarizing, here we have considered the system with two non-gravitating coupled scalar fields in cartesian coordinates. For such a system, as well as in general relativity, the task of finding regular solutions amounts to searching eigenvalues of the parameters of the model. The model contains six available parameters: two initial values of the scalar fields and four free parameters $\Lambda_1, \Lambda_2, m_1, m_2$ (Λ can be always excluded by redefinition of the parameters Λ , and the initial values of derivatives φ ((0) are chosen to be equal to zero for obtaining symmetric solutions). Then for obtaining regular solutions it is necessary to find eigenvalues of only two of these six parameters. For example, we have been sought the eigenvalues of the parameters μ_1 and μ_2 in the system (6)-(9). For the obtained eigenvalues regular solutions start at $x = 0$ and tend to the fixed point A corresponding to the local minimum of the system (for the case of usual scalar fields considered here, see Figs. 1, 4 and 5), and to the local maximum (for phantom/ghost fields). One can see from Fig. 3 that there exist some lowest eigenvalues of the parameters μ and μ_2 at which the total energy of the system M goes to zero. This corresponds to the existence of

some critical μ_1 and μ_2 at which physically sensible solutions with a nonzero total energy still exist. Similarly, for the system (18)-(21) there exists some critical eigenvalue of the parameter μ at which $M \to 0$ as well (see Fig. 6).

Type "unstable node" fixed points allow the existence of sets of solutions starting from these points at $x = \pm\infty$ both for usual fields (the fixed points A, B and C, D) and for phantom/ghost scalar fields (the point E). Thus the qualitative analysis shows that from the point of view of a possibility of obtaining regular localized solutions the system with two coupled scalar fields in question seems to be quite perspective. The previous studies from Refs. [9]-[17] show that inclusion of gravitational fields does not change the qualitative behavior of solutions. Then one would expect, for example, that in the presence of gravitational fields the lower restriction on the values of the parameters μ_1 and μ giving physically sensible solutions will also exist.

Thus we have investigated two coupled autonomous Gingburg – Landau equations as eigenvalue problem, found all critical points and the corresponding phase portraits are found.

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