OSCILLATING INSTABILITY IN A FINITE SIZE CYLINDRICAL CHANNEL

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The problem of stability of mixing in a ternary isothermal mixture is solved for a finite-size diffusion channel using the perturbation method. We provide a comparative analysis between the positions of the oscillating stability line for a plane infinite layer and for 'the mass permeability' of a cylindrical channel of a finite size. Our theoretical results are in a good agreement with the experimental data obtained by the two-flasks method for the N_2*–0.333He+0.667Ar* system.

Of great significance in nature and engineering are mass flows caused by the interaction of two thermodynamic forces. This mass transfer is called convection which results from double diffusion [1]. It was shown experimentally and analytically [2] that 'cross rolls' give rise to drastic oscillating processes, which lead to absolute instability in some regions of a uniform flow of fluids. The regions of stable and unstable flows of fluids on the plane of Reynolds and Rayleigh numbers were determined for oil and water. Heat exchange processes during melting of ice in the Arctic Regions were studied in [3]. At a temperature of 0°C the freshwater layer, which is formed from ice, underlies the seawater layer. It is known that the density of freshwater is higher than that of seawater, therefore this system is unstable. The authors of [3] noted that the density gradient is inverted at large Rayleigh numbers and the heat exchange rate increases by a factor of approximately five to ten.

Numerical studies on the heat transfer by natural convection in a square cavity, which was performed by using the method of discrete ordinates, have demonstrated that some parameters (as the Rayleigh numbers, the buoyancy index, the Lewis numbers and the optical thickness of the liquid) influence the flow structure [4]. It was shown that given certain parameters of the system one can observe the heat transfer caused by natural convection, double diffusion or irradiation. An analysis of the stability of a ferrofluid heated from below in a homogeneous vertical magnetic field has showed that a convective instability occurs at a critical temperature gradient [5]. The parameters, at which an oscillating instability arises, were determined by the Galerkin method for a magnetoconductive fluid layer between two free boundaries compressing the surface [5].

A study of diffusion in isothermal ternary gas systems with a stable stratification of the density showed that an instability of the mechanical equilibrium and subsequent development of a convective process take place in some systems depending on the thermodynamic parameters (e.g. the pressure, the temperature and the concentration of components) and the geometrical dimensions of the diffusion channel (the length and the characteristic size) [6, 7]. In the literature this phenomenon is referred to as instability of the mechanical equilibrium [6], diffusion instability [1], or 'double diffusion' [8]. These studies demonstrated that the diffusion process is replaced by a convective process of various intensities at a certain value of the thermodynamic parameter, mainly the pressure and/or the concentration.

This paper deals with the process of mixing in a gas system comprised of helium, argon, and nitrogen. The gas system was chosen such that the density of the overlying gas was equal to the density of the underlying mixture. The experimental studies were performed by a two-flask method, which is widely used to analyze characteristics of diffusion processes [9–11]. A binary gas mixture of 0.333He+0.667Ar is placed in the bottom flask of the diffusion apparatus, while pure nitrogen is placed in the top flask. The diffusion channel has a form of a cylinder of length L=6 cm and of radius r=2 mm. The temperature of the flasks T=298K and the experimental time t=20 min are kept constant. The experimental pressure varies in the range from 0.5 MPa to 4.0 MPa.

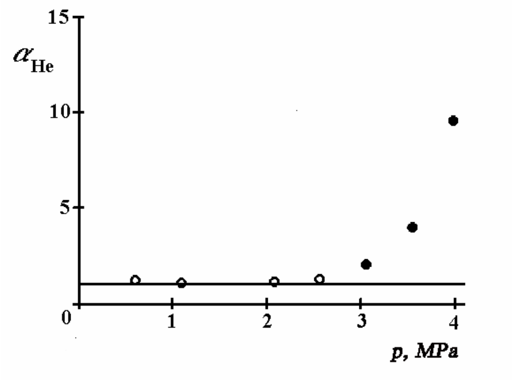

Fig. 1. Dependence of the parameter α of helium on the pressure

The results of measurements performed earlier are given in figure 1 as the pressure dependence of the dimensionless parameter α of helium, which is the most lightweight component of the mixture. The parameter α is defined as the ratio between the experimental and calculated concentrations assuming the diffusion mixing. One can see from figure 1 that the diffusion transfer takes place in the system up to the pressure of ≈2.6 MPa, as it follows from an agreement of the experimental and theoretical data, i.e. α=1 within the statistical errors. The further increase of the pressure leads to the instability of mechanical equilibrium of the gas mixture. This results in a drastic discrepancy of the observed concentration from the theoretical value, meaning that α>1. We would like to explain this discrepancy and to find the conditions, under which the oscillating instability appears in a ternary mixture.

The class of problems related to the concentration isothermal convection, particularly the motion of a ternary gas mixture in the presence of a spatial inhomogeneity caused by a non-uniformity of the densities in the gravity field is described by a set of equations of fluid dynamics. They include Navier--Stokes equations of motion and the two equations for conservation of the number of particles in the mixture and for the components, respectively [7, 12]:

$$\rho\left[\frac{\partial\vec{u}}{\partial t}+(\vec{u}\nabla)\vec{u}\right]=-\nabla p+\eta\Delta\vec{u}+\left(\frac{\eta}{3}+\xi\right)\nabla div\vec{u}+\rho\vec{g},$$
$$\frac{\partial n}{\partial t}=-div(n\vec{v}),$$
$$\frac{\partial c_1}{\partial t}+\vec{v}\nabla c_1=div[D_{11}^*\nabla c_1+D_{12}^*\nabla c_2], \tag{1}$$
$$\frac{\partial c_2}{\partial t}+\vec{v}\nabla c_2=div[D_{21}^*\nabla c_1+D_{22}^*\nabla c_2],$$

where $\vec{u}=\dfrac{\rho_1\vec{u}_1+\rho_2\vec{u}_2+\rho_3\vec{u}_3}{\rho}$ and $\vec{v}=\dfrac{n_1\vec{u}_1+n_2\vec{u}_2+n_3\vec{u}_3}{n}$ are the mass-average and the number-average velocities, respectively; ρ is the mixture density; p is the pressure; c_i is the concentration of the i-th component (the mixture components are ordered in a such way that $m_1<m_3<m_2$, with m_i being the molecular mass of the i-th component); $\vec{g}$ is the gravitational acceleration; η (ξ) are coefficients of the shift (bulk) viscosities; D_{ij}^* denotes the 'observed' three-component diffusion constants.

The set of equations (1) are supplemented by the general state equation of the environment:

$$\rho=\rho(c_1,c_2,p),\ T=const. \tag{2}$$

To solve the set of equations (1)–(2), it is necessary to specify the boundary conditions. We assume therefore that the length of the cylindrical channel L along the z-axis (figure 2) is much larger than its radius r.

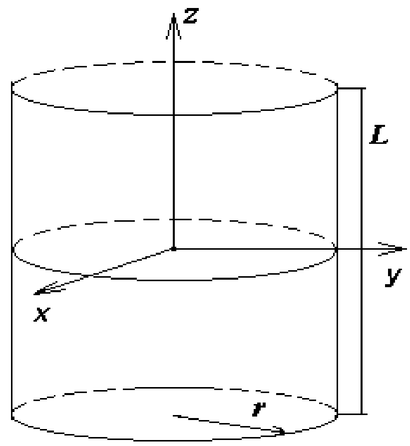

Fig. 2. Vertical cylindrical diffusion channel

In this case, the forces ∇p and $\rho\vec{g}$ act only in the direction of the z-axis. Therefore, we are interested only in the projection of the Navier-Stokes equation to the z-axis:

$$v_x = v_y = 0, \qquad u_x = u_y = \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \qquad u_z = u(x), v_z = v(x).$$

Comparing the terms of this equation for $L >> 2r$ and neglecting the terms proportional to $\frac{r^2}{L^2}$, we have the following equation:

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial z} + \eta\frac{\partial^2 u}{\partial x^2} + \rho g_z\,. \tag{3}$$

In the case of steady-state mixing of the gases in the absence of free convection $\rho = \rho_0(z)$, $p = p_0(z)$, the equation (3) takes the form

$$\left(\frac{\partial p_0}{\partial z} - \rho_0 g_z\right) = \eta\frac{\partial^2 u_0}{\partial x^2}\,. \tag{4}$$

Integrating (4) leads to a classical parabolic profile for the velocity

$$u_0 = \frac{x^2 - r^2}{2\eta}\left(\frac{\partial p_0}{\partial z} - \rho_0 g_z\right). \tag{5}$$

Subtracting (4) from (3), we have perturbation equations:

$$\frac{\partial u'}{\partial t}=\frac{\eta}{\rho}\frac{\partial^2 u'}{\partial x^2}+g_z(\beta_1 c_1{}' +\beta_2 c_2{}'), \tag{6}$$

where $\beta_i=\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial c_i}\right)_{p,T,c_j}$.

In rearranging from (3) to (6) we neglect possible pressure perturbations, because the pressure relaxation time is much shorter than the concentration relaxation time.

Let us consider the equation for convective diffusion in the set of equations (1). The steady-state diffusion in the vertical channel with the induced convection is described by the following equation

$$v_0\frac{\partial c_{10}}{\partial z}=\frac{\partial}{\partial z}\left[D_{11}^*\frac{\partial c_{10}}{\partial z}+D_{12}^*\frac{\partial c_{20}}{\partial z}\right]. \tag{7}$$

Taking into account that the concentration is independent of the transverse coordinate and subtracting (7) from (1) we obtain the following perturbation equation:

$$\frac{\partial c_1'}{\partial t}+v'\frac{\partial c_{10}}{\partial z}=D_{11}^*\frac{\partial^2 c_1'}{\partial x^2}+D_{12}^*\frac{\partial^2 c_2'}{\partial x^2}\,. \tag{8}$$

Consider that

$$\left(u'-v'\right)=D_1\frac{\partial c_1'}{\partial z}+D_2\frac{\partial c_2'}{\partial z}, \tag{9}$$

where D_1 and D_2 are some quantities, which have dimensionality and values on the order of the diffusion constant, it easy to understand that the differences between the corrections of the mass-average velocity and the number-average velocity is insignificant in the equation (8). Therefore, v' in the above equations can be substituted by u'.

Thus, the set of equations for perturbations of the velocity u' and the concentrations c_i' in the vertical channel becomes as follows:

$$\begin{gathered}\frac{\partial u'}{\partial t}=\frac{\eta}{\rho}\frac{\partial^2 u'}{\partial x^2}+g_z(\beta_1 c_1' +\beta_2 c_2'),\\ \frac{\partial c_2'}{\partial t}+u'\frac{\partial c_{10}}{\partial z}=D_{11}^*\frac{\partial^2 c_1'}{\partial x^2}+D_{12}^*\frac{\partial^2 c_2'}{\partial x^2}\,,\\ \frac{\partial c_2'}{\partial t}+u'\frac{\partial c_{20}}{\partial z}=D_{21}^*\frac{\partial^2 c_1'}{\partial x^2}+D_{22}^*\frac{\partial^2 c_2'}{\partial x^2}\,,\\ \frac{\partial u'}{\partial z}=0\,.\end{gathered} \tag{10}$$

Let us solve the set of equation (10). Taking the characteristic dimension of the problem to be equal to the channel radius r and rendering (10) dimensionless, we have (omitting the primes):

$$P_{22}\frac{\partial c_1}{\partial t}-u=\tau_{11}\frac{\partial^2 c_1}{\partial x^2}+\frac{A_2}{A_1}\tau_{12}\frac{\partial^2 c_2}{\partial x^2},$$
$$P_{22}\frac{\partial c_2}{\partial t}-u=\frac{A_1}{A_2}\tau_{21}\frac{\partial^2 c_1}{\partial x^2}+\frac{\partial^2 c_2}{\partial x^2},\tag{11}$$
$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+R_1\tau_{11}c_1+R_2c_2,$$
$$\frac{\partial u}{\partial z}=0,$$

where $P_{ii}=\frac{\nu}{D_{ii}^*}$ is the Prandtl diffusion number; $R_i=\frac{g\beta_1\Delta c_i d^4}{\nu D_{ii}^* L}$ is the Rayleigh partial number for the *i*-th component; ν is the kinematic viscosity; $\tau_{ij}=\frac{D_{ij}^*}{D_{22}^*}$ denotes the parameters, which determine the relationship between the 'observed' diffusion coefficients: $\nabla c_{i0}=-A_i\vec{\gamma}$.

To determine the criticality conditions for the oscillating instability, the solution of the set of equations (11) is sought in the form

$$u_z(x,y,z)=u_0\exp[-\lambda t]\sin\left(n\frac{\pi}{2}x\right)\sin\left(n\frac{\pi}{2}y\right)\sin\left(n\frac{\pi}{4}z\right),$$
$$c_k(x,y,z)=c_i\exp[-\lambda t]\sin\left(n\frac{\pi}{2}x\right)\sin\left(n\frac{\pi}{2}y\right)\sin\left(n\frac{\pi}{4}z\right),\tag{12}$$

where *n=1,3,5,…* stands for characteristic odd modes of perturbations, and λ is the perturbation time decrement. The boundary conditions assume that the velocity perturbations and the flow of matter on the walls of the channel become zero:

$$u=0,\ \frac{\partial c_i}{\partial x}=0,\quad x=\pm r.$$

Substituting (12) into the set of equations (11) and excluding sequentially the amplitudes of the concentrations and the velocity subsequently, we obtain a cubic equation with respect to λ. This expression determines characteristic roots for any *n* depending on the Rayleigh and Prandtl numbers, the concentration gradients in the following form:

$$p\lambda^3+q\lambda^2+r\lambda+s=0,\tag{13}$$

where

$$p=P_{22}^2,$$
$$q=P_{22}\left[(n+1)\frac{\pi}{2}\right]^2\left(-P_{22}-1-\tau_{11}\right),$$
$$r=\left[(n+1)\frac{\pi}{2}\right]^4\left\{P_{22}(1+\tau_{11})+\tau_{11}-\tau_{12}\tau_{21}\right\}+P_{22}\left(R_1+\tau_{11}+R_2\right),$$

$$s=\left[(n+1)\frac{\pi}{2}\right]^6\left[\tau_{12}\tau_{21}-\tau_{11}\right]+\left[(n+1)\frac{\pi}{2}\right]^2\left[\left(1-\frac{A_2}{A_1}\tau_{12}\right)R_1\tau_{11}+\left(\tau_{11}-\frac{A_1}{A_2}\tau_{21}\right)R_2\right].$$

Depending on the values of *p, q, r,* and *s*, the equation (13) gives either three real roots (monotonic perturbations) or one real and two complex conjugate roots (the latter describe oscillatory perturbations). If the decrement of the cubic equation (13) has a nonzero imaginary part $(\lambda=\lambda_r+i\omega)$, the perturbation terms oscillate at a frequency equal to the imaginary part of the decrement ω; attenuation (rise) of the perturbations is determined by the sign of λ_r. The loss of stability of the gas mixture means that the decrements λ of some characteristic perturbations at a certain number R_* (or numbers R_{n*}) reverse sign. The perturbations start growing at $R>R_*$, while they are decaying at $R<R_*$. Vanishing of the decrement λ determines the condition that the perturbation is "neutral" meaning that it neither rises nor decays. This condition characterizes the boundary of the equilibrium stability with respect to a given perturbation.

Setting $\lambda_r=0$ for determination of the stability boundaries, the equation (13) can be rewritten in the form of the following expressions:

$$-q\omega^2+s=0, \tag{14}$$

$$\omega\left(-p\omega^2+r\right)=0. \tag{15}$$

The latter equations determine the critical conditions for the Rayleigh partial numbers and the perturbation frequency ω at the boundary of oscillating stability. Equating the brackets in (15) to zero, from (14) and (15) it is possible to deduce an equation for the oscillating instability line:

$$\tau_{11}\left(\frac{A_2}{A_1}\tau_{12}-P_{22}-\tau_{11}\right)R_1+\left(\frac{A_1}{A_2}\tau_{21}-P_{22}-1\right)R_2= \tag{16}$$

$$=\left((n+1)\frac{\pi}{2}\right)^4\left[\frac{1}{P_{22}}\left\{P_{22}(1+\tau_{11})+\tau_{11}-\tau_{12}\tau_{21}\right\}\left[-P_{22}-1-\tau_{11}\right]-\tau_{21}\tau_{12}-\tau_{11}\right].$$

The frequency of neutral oscillations is defined by the formula

$$\omega^2=\frac{s}{q}=\frac{\left[(n+1)\frac{\pi}{2}\right]^4\left[\tau_{12}\tau_{21}-\tau_{11}\right]+\left(1-\frac{A_2}{A_1}\tau_{12}\right)R_1\tau_{11}+\left(\tau_{11}-\frac{A_1}{A_2}\tau_{21}\right)R_2}{P_{22}\left(-P_{22}-1-\tau_{11}\right)}, \tag{17}$$

in which the Rayleigh numbers R_1 and R_2 satisfy the condition (16). It should be noted that the straight line defined by (16) has the meaning of a neutral line for oscillatory perturbations only in the $\omega^2>0$-section.

Figure 3 shows the mutual positions of the oscillating (*KK*) and monotonic (*MM*) stability lines for the (N_2–0.333He+0.667Ar)-system studied experimentally (cf figure 1). The points standing for the experimental data were calculated assuming a nonlinear distribution of the concentration along the length of the finite-size diffusion channel:

$$R_1=\frac{gnr^4\Delta m_1}{\rho\nu D_{11}^*}\cdot\frac{\partial c_1}{\partial z},\quad R_2=\frac{gnr^4\Delta m_2}{\rho\nu D_{22}^*}\cdot\frac{\partial c_2}{\partial z}. \tag{18}$$

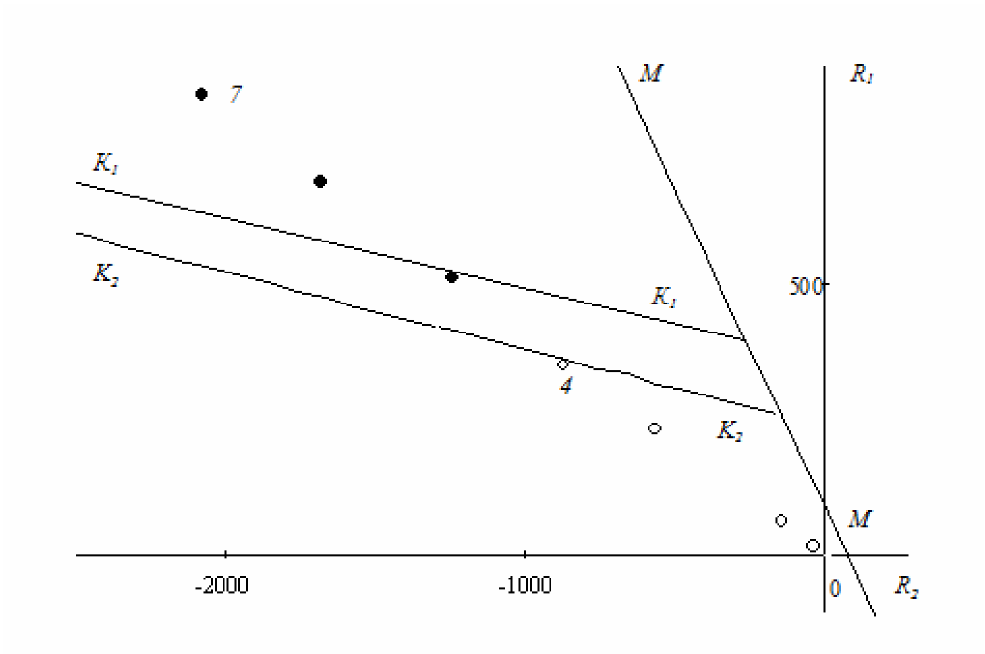

Fig. 3. Stability lines for the (N_2–0.333He+0.667Ar)-system. The points correspond to the pressure values of (1) 0.584, (2) 1.074, (3) 2.06, (4) 2.55, (5) 3.04, (6) 3.53, and (7) 3.92 MPa, respectively

It is seen from figure 3 that the experimental points corresponding to the diffusion lie below the lines of the monotonic and oscillating stabilities. The points corresponding to the convective mass transfer are located above the line K_1K_1, but below the line *MM*. This is an indication that this system has an oscillating instability of the mechanical equilibrium. A comparison of the oscillating instability line calculated for a flat vertical layer (K_2K_2) [12] shown in figure 3 and the line calculated for the 'mass-impermeable' cylindrical channel (K_1K_1) on the basis of the proposed model shows that the "diffusion–convection" transition boundary are in a better agreement with the experimental data.

In summary, we have shown that an oscillating instability in a gravity field appears in the three-component mixture when the pure component is at the top and the binary mixture is on the bottom of the diffusion channel. Secondly, if the length of the diffusion cylindrical channel is finite and the concentration is nonlinearly distributed along the channel our results reflect properly the experimental conditions. Finally, the position of the line of the oscillating instability calculated for a 'mass-impermeable' cylindrical channel agrees well with the earlier measurements. This agreement is much better than that for the simplest case of a two-dimensional infinite vertical layer.

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ШЕКТЕЛГЕН ЦИЛИНДРЛІК КАНАЛДАҒЫ ТЕРБЕЛМЕЛІ ОРНЫҚСЫЗДЫҚ

И.В. Поярков

Ең аз әсер әдісін қолдана отырып шектелген диффузиялық каналда изотермдік үшкомпоненттік орнықты араласудың есебі шешілді. Шектелген масса өткізбейтін цилиндрлік канал мен жазық шексіз қабаттар үшін тербелмелі орнықтылық сызығына салыстырмалы талдаулар жасалды. Екіколбалық әдіс арқылы N_2 – $0.333He$ + $0.667Ar$ жүйесіне жүргізілген теориялық есептеулер нәтижелері эксперимент мәндерімен жақсы сәйкес келеді.

КОЛЕБАТЕЛЬНАЯ НЕУСТОЙЧИВОСТЬ В ОГРАНИЧЕННОМ ЦИЛИНДРИЧЕСКОМ КАНАЛЕ

И.В. Поярков

Методом малого параметра решена задача об устойчивости процесса смешения в тройной изотермической смеси для ограниченного диффузионного канала. Дано сравнение положение линии колебательной устойчивости для плоского слоя и массонепроницаемого цилиндрического канала конечных размеров. Теоретические результаты исследования сопоставлены с опытными данными, полученными двухколбовым методом для системы N_2 – $0,333He$ + $0,667Ar$.