IRSTI 29.35.03, 27.35.55 https://doi.org/10.26577/RCPh20994126

Z.K. Shanina[1*], N. Serikbayev[2]
[1]K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan
[2]L.N. Gumilyov Eurasian National University, Astana, Kazakhstan
*e-mail: zshanina@zhubanov.edu.kz

# HAMILTONIAN STRUCTURE AND SOLITONS FOR KONNO-OONO EQUATION WITH SELF-CONSISTENT SOURCES

In this paper, we investigate the system of Konno-Oono equation with self-consistent sources. The aim of this research is to determine the conserved quantities (Hamiltonians) from the Riccati equations for the Konno- Oono system with self-consistent sources and to obtain one-soliton solutions using Hirota's bilinear method. This model provides an idealized description of how short pulses propagate in magnetic fields and in various physical contexts, including nonlinear waves, optics, and hydrodynamics. We obtain Lax pairs and an integrability condition for systems of Konno - Oono equations. We derive its Hamiltonian structure by expanding the spectral parameter into inverse powers, resulting in an infinite hierarchy of conserved quantities expressed in terms of smooth real functions and their derivatives. We use the Hirota bilinear method to construct soliton solutions. By introducing transformations of the dependent variables and expanding them into power series, we obtain one-soliton solutions for the fields $q$, $\rho$, $p$, and $\eta$. We construct graphical representations of localized stable wave profiles characteristic of solitons in such systems.

**Keywords:** conserved quantities, soliton solution, Hirota bilinear method, dispersionless equation, integrable system, localized structure.

З.К. Шанина[1]*, Н. Серікбаев[2]
[1]Қ. Жұбанов атындағы Ақтөбе өңірлік университеті, Ақтөбе, Қазақстан
[2]Еуразия ұлттық университеті, Астана, Қазақстан
*e-mail: zshanina@zhubanov.edu.kz

# Өзіндік үйлесімді көздері бар Конно-Уно теңдеуінің гамильтондық құрылымы және солитондық шешімдері

Бұл ұсынылған мақалада өздігінен үйлесімді көздері бар Конно - Уно теңдеулер жүйесі зерттелді. Зерттеудің мақсаты өздігінен үйлесімді көздері бар Конно - Уно жүйесі үшін Рикатти теңдеулерінен сақталатын шамаларды (гамильтонтандарды) анықтау және Хиротаның бисызықтық әдісін қолдана отырып, бір солитондық шешімдерді алу болып табылады. Бұл қарастырылған модель магнит өрістерінде және әртүрлі физикалық контексттерде, соның ішінде сызықтық емес толқындарда, оптикада және сұйықтар динамикасында қысқа импульстардың таралуының идеалдандырылған сипаттамасын береді. Конно - Уно теңдеулерінің жүйесі үшін Лакс жұптары және интегралдану шарттары алынды. Оның гамильтондық құрылымы спектрлік параметрді кері дәрежелерде кеңейту арқылы алынды, және бұл тегіс нақты функциялар және олардың туындылары арқылы өрнектелген сақталатын шамалардың шексіз иерархиясына әкелді. Сондай ақ, солитондық шешімдерді құру үшін Хирота бисызықтық әдісі қолданылды. Тәуелді айнымалыларды түрлендіруді енгізу және оларды дәрежелік қатарларда кеңейту арқылы $q$, $\rho$, $p$ және $\eta$ өрістері үшін бір солитондық шешімдер алынды. Мұндай жүйелердегі солитондарға тән тұрақты толқындардың локализацияланған профильдерінің графикалық көріністері құрылды.

**Түйін сөздер:** сақталатын шамалар, солитондық шешім, Хирота бисызықтық әдісі, дисперсиясыз теңдеу, интегралданатын жүйе, локализацияланған құрылым.

 © 2026 Al-Farabi Kazakh National University

З. К. Шанина[1]*, Н. Серикбаев[2]
[1]К. Актюбинский региональный университет имени К. Жубанова, Актобе, Казахстан
[2]Евразийский национальный университет, Астана, Казахстан
*e-mail: zshanina@zhubanov.edu.kz

# Гамильтонова структура и солитонные решения уравнения Конно-Уно со самосогласованными источниками

В данной работе исследована система уравнений Конно - Уно с самосогласованными источниками. Цель исследования данной работы является определение сохраняющихся величин (гамильтониан) из уравнений Риккати для системы Конно - Уно с самосогласованными источниками и получение односолитонных решений с использованием билинейного метода Хироты. Эта модель обеспечивает идеализированное описание распространения коротких импульсов в магнитных полях и в различных физических контекстах, включая нелинейные волны, оптику и гидродинамику. В работе были получены пары Лакса и условие интегрируемости для систем уравнений Конно - Уно. Была выведена его гамильтонова структура путем разложения спектрального параметра в обратные степени, что привело к бесконечной иерархии сохраняющихся величин, выраженных через гладкие вещественные функции и их производные. Был также использован билинейный метод Хироты для построения солитонных решений. Путем введения преобразований зависимых переменных и разложения их в степенные ряды получены односолитонные решения для полей $q$, $\rho$, $p$ и $\eta$. Построены графические представления локализованных профилей устойчивых волн, характерные для солитонов в таких системах.

Ключевые слова: сохраняющиеся величины, солитонное решение, билинейный метод Хироты, бездисперсионное уравнение, интегрируемая система, локализованная структура.

## Introduction

Integrable systems or so-called solvable nonlinear differential equations have become widely used in mathematics and physics after the discovery of the inverse scattering method. By integrable systems we mean those that have an infinite hierarchy of symmetries and conservation laws.

In addition to integrable nonlinear differential equations, there is another important class of integrable equations, often called dispersionless equations or hydrodynamic type equations. Equations of this type have been actively studied in recent years due to their appearance in various problems of physics and mathematics [1-8].

Some of these equations are integrable in the Hamiltonian sense. Hamiltonians play an important role in the field of integrable systems, and they also have a fundamental place in other areas of physics and mathematics [9-15]. For this reason, Hamiltonians have found a remarkable number of applications in both physics and mathematics. They are often called the principal integrable system.

Although the Konno–Oono coupled dispersionless system was initially introduced as an integrable mathematical model, subsequent studies have shown that it can describe several physical phenomena. This model is used as an idealized model for the propagation of short pulses in magnetic fields, nonlinear waves, optics, and hydrodynamics. In the current literature, the Konno–Oono system is often considered as a model for describing current-field strings interacting with an external magnetic field. The solitons obtained by the Konno–Oono equation are used to describe directional waves in liquid crystals. In nematic crystals, the orientation of molecules changes in space. These changes propagate in the form of waves. The Konno–Oono model can describe the propagation of such nonlinear waves [16–24]. In addition, some forms of the Konno–Oono system are related to the Wadati–Konno–Ichikawa (WKI) spectral problem [2,3]. WKI systems are used to characterize short optical pulses, ultrashort laser waves, and optical solitons. A short light pulse propagating through an optical fiber is preserved as a soliton as a result of the balance of dispersion and nonlinearity.

Dispersionless systems are often associated with hydrodynamic-type equations. The dispersionless limit of the Konno–Oono equation can be used for shallow water waves, hydrodynamic reductions, and quasiclassical flows. The generalized Konno-Oono

equation with self-consistent sources was studied in [25–26]. In these studies, the dispersionless limit and soliton surfaces were constructed, the Gaussian and mean curvatures were determined, and the surface area was calculated for an integrable equation without dispersion with self-consistent sources in Riemannian space. Darboux transformations and soliton solutions for the generalized Koono-Oono equation were obtained by the authors [27].

In this paper, soliton solutions are considered using the Hirota method for the generalized Koono-Oono equation with self-consistent sources. From a methodological point of view, the Hirota method remains the main tool for constructing explicit solutions.

**Materials and Methods**

The main formulas of the Konno-Oono equation with self-consistent sources (KOESCS) reads as [1]

$$q_{xt} - 4aq + 2p_x = 0, \quad (1)$$

$$r_{xt} - 4ar - 2n_x = 0, \quad (2)$$

$$a_x - 0{,}5(rq)_t + qn - rp = 0, \quad (3)$$

$$\eta_x + rp - qn = 0, \quad (4)$$

$$p_x + 2i\omega p + 2\eta q = 0, \quad (5)$$

$$n_x - 2i\omega n - 2\eta r = 0. \quad (6)$$

This system is integrable. Its Lax representation (LR) can be expressed as

$$\Psi_x = U_2\Psi, \quad (7)$$

$$\Phi_t = V_2\Psi. \quad (8)$$

Here

$$U_2 = -i\lambda\sigma_3 + Q, V_2 = \frac{i}{\lambda}F + \frac{i}{\lambda-\omega}G,$$

where

$$Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, F = \begin{pmatrix} a & -0.5q_t - p \\ 0.5r_t - n & -a \end{pmatrix},$$

$$G = \begin{pmatrix} \eta & p \\ n & -\eta \end{pmatrix}.$$

Note that from equations (7)-(8) we have the following important formula

$$\eta^2 + np = const$$

or for simplicity

$$\eta^2 + np = 1.$$

Consider the following scale transformation

$$a \to 0,25p, (q,r) \to 0,5(q,r).$$

Then the KOESCS takes the form

$$q_{xt} - \rho q + 4p_x = 0, \quad (9)$$

$$r_{xt} - \rho r - 4n_x = 0, \quad (10)$$

$$\rho_x - 0{,}5(rq)_t + 2(qn - rp) = 0, \quad (11)$$

$$\eta_x + 0{,}5(rp - qn) = 0, \quad (12)$$

$$p_x + 2i\omega p + \eta q = 0, \quad (13)$$

$$n_x - 2i\omega n - \eta r = 0. \quad (14)$$

In the case $p = n = \eta = 0$ system (9)-(14) takes the form

$$q_{xt} - \rho q = 0,$$

$$r_{xt} - \rho r = 0,$$

$$\rho_x - 0{,}5(rq)_t = 0,$$

which is a more standard form of the generalized Konno–Oono equation.

**Hamiltonian Structure**

From equations (7)-(8), we get the following scattering problem:

$$\Phi_{1x} = -i\lambda\Phi_1 + q\Phi_2 \; \Phi_{2x} = r\Phi_1 + i\lambda\Phi_2 \quad (15)$$

and

$$\Phi_{1t} = \left(\frac{i}{\lambda}a + \frac{i}{\lambda-\omega}\eta\right)\Phi_1 + \left(\frac{i}{\lambda}(-0{,}5q_t - p) + \frac{i}{\lambda-\omega}p\right)\Phi_2 \quad (16)$$

$$\Phi_{2t} = \left(\frac{i}{\lambda}(0{,}5r_t - n) + \frac{i}{\lambda-\omega}n\right)\Phi_1 + \left(-\frac{i}{\lambda}a - \frac{i}{\lambda-\omega}\eta\right)\Phi_2.$$

The integrability conditions read

$$U_{2t} - V_{2x} + [U_2, V_2] = 0,$$

or in component form as the set of equations (1)-(6). We introduce the quantity

$$\Gamma = \frac{\Phi_2}{\Phi_1}. \quad (17)$$

Differentiating equation (17) with respect to $x$ and $t$, and using equations (15), (16) and (17), then equations (7)-(8) reduce to the Riccati equations:

$$\Gamma_x = r + 2i\lambda\Gamma - q\Gamma^2, \quad (18)$$

$$\Gamma_t = \frac{i}{\lambda}(0{,}5r_t - n) + \frac{i}{\lambda-\omega}n - 2\Gamma\left(\frac{i}{\lambda}a + \frac{i}{\lambda-\omega}\eta\right) + \Gamma^2\left(\frac{i}{\lambda}(0{,}5q_t + p) - \frac{i}{\lambda-\omega}p\right).$$

From the relation

$$\Gamma_{xt} = \Gamma_{tx}$$

we get

$$(q\Gamma)_t = \left(\frac{i}{\lambda}a_x + \frac{i}{\lambda-\omega}\eta_x + \Gamma\left(\frac{i}{\lambda-\omega}p - \frac{i}{\lambda}(0{,}5q_t + p)\right)\right)_x,$$

where the expressions on the left-hand side represent the conserved densities, whereas those on the right-hand side represent the corresponding fluxes. From equation (15) we obtain

$$\frac{\Phi_{2x}}{\Phi_2} = r\Gamma + i\lambda,$$

then

$$\ln\Phi_2 = \int r\Gamma dx + i\lambda x. \quad (19)$$

Rearranging equation (18), we write it as

$$2i\lambda r(r\Gamma) = r^2\left(\frac{r\Gamma}{r}\right)_x - r^3 + q(r\Gamma)^2. \quad (20)$$

Then, we expand $r\Gamma$ into a power series in inverse power of $\lambda$ as follows

$$r\Gamma(x,t,\lambda) = \sum_{n=1}^{\infty}\phi_n(x,t)\lambda^{-n}. \quad (21)$$

In the same way, we expand $\ln\Phi_2$ into a power series in inverse power of $\lambda$ so that

$$\ln\Phi_2 = H_0 + \sum_{l=1}^{\infty}H_l\lambda^{-l}. \quad (22)$$

Here $H_0$ and $H_l$ are Hamiltonians (conserved quantities). By substituting (21) into (20) we obtain the following system of conservation laws

$$2i\lambda r\sum_{n=1}^{\infty}\phi_n\lambda^{-n} = r^2\left(\frac{\sum_{n=1}^{\infty}\phi_n\lambda^{-n}}{r}\right)_x - r^3 + q(\sum_{n=1}^{\infty}\phi_n\lambda^{-n})^2.$$

Next we get

$$\phi_1 = \frac{i}{2}r^2,$$

$$\phi_2 = \frac{1}{4}rr_x,$$

$$\phi_3 = -\frac{i}{2}\left(r\left(\frac{\phi_2}{r}\right)_x + \frac{q}{r}\phi_1^2\right),$$

$$\phi_{n+1} = -\frac{i}{2}\left(\frac{q}{r}\sum_{k=1}^{n-1}\phi_k\phi_{n-k} + r\left(\frac{\phi_n}{r}\right)_x\right), n \geq 3.$$

Then, by equating powers of $\lambda$ on both sides of Equations (19), (21) and (22), we obtain an infinite number of Hamiltonians, which can be explicitly determined in terms of smooth real functions $\Phi_n(x,t)$ and their derivatives, as follows

$$H_0 = i\lambda x,$$

$$H_1 = \frac{i}{2}\int r^2 dx,$$

$$H_2 = \frac{1}{4}\int rr_x dx,$$

$$H_3 = -\frac{i}{2}\int (rr_{xx} - qr^3)dx,$$

$$H_{n+1} = -\frac{i}{2}\int\left(\frac{q}{r}\sum_{k=1}^{n-1}\Phi_k\Phi_{n-k} + r\left(\frac{\Phi_n}{r}\right)_x\right)dx,$$

$n \geq 3$.

**Bilinear forms and soliton solutions**

If $r = \bar{q}, n = \bar{p}$, then the system of equations (9)-(14) can be reduced to

$$q_{xt} - \rho q + 4p_x = 0, \quad (23)$$

$$\rho_x - 0{,}5(\bar{q}q)_t + 2(q\bar{p} - \bar{q}p) = 0, \quad (24)$$

$$\eta_x + 0{,}5(\bar{q}p - q\bar{p}) = 0, \quad (25)$$

$$p_x + 2i\omega p + \eta q = 0. \quad (26)$$

Certain soliton solutions for the nonlinear evolution equations can be constructed via the Hirota bilinear method. In the following, we will use this method to obtain multisoliton solutions for equations (23)-(26). Similar to Maxwell – Bloch equations, $p$ and $\eta$ can be given by [1]

$$p = 2\psi_1\bar{\psi}_2, n = -2\sigma\bar{\psi}_1\psi_2, \quad \eta = |\psi_1|^2 + \sigma|\psi_2|^2, \tag{27}$$

where $|\psi_1|^2 - \sigma|\psi_2|^2 = const$ and $\Psi = (\psi_1, \psi_2)^T$ satisfy the equations (14)-(15). Then the system of equations (23)-(26) takes the form

$$q_{xt} - 4aq - 8i\xi\psi_1\bar{\psi}_2 + 4q(\sigma|\psi_1|^2 + |\psi_2|^2) = 0, \tag{28}$$

$$a_x - 0{,}5\sigma(|q|^2)_t - 2\sigma(q\bar{\psi}_1\psi_2 + \bar{q}\psi_1\bar{\psi}_2) = 0, \tag{29}$$

$$\psi_{1x} + i\xi\psi_1 - q\psi_2 = 0, \tag{30}$$

$$\psi_{2x} - i\xi\psi_2 - \sigma\bar{q}\psi_1 = 0. \tag{31}$$

Through the transformations

$$\psi_1 = e^{i\xi x}a_1, \psi_2 = e^{i\xi x}a_2, \bar{\psi}_1 = e^{-i\xi x}\bar{a}_1, \bar{\psi}_2 = e^{-i\xi x}\bar{a}_2,$$

we can rewrite equation (28)-(31) as

$$q_{xt} - 4aq - 8i\xi a_1\bar{a}_2 + 4q(\sigma|a_1|^2 + |a_2|^2) = 0, \tag{32}$$

$$a_x - 0{,}5\sigma(|q|^2)_t - 2\sigma(q\bar{a}_1a_2 + \bar{q}a_1\bar{a}_2) = 0, \tag{33}$$

$$a_{1x} + i\xi a_1 - qa_2 = 0, \tag{34}$$

$$a_{2x} - i\xi a_2 - \sigma\bar{q}a_1 = 0. \tag{35}$$

Consider the following transformation

$$a = \frac{\rho}{4}, \sigma = -1,$$

Then equations (32)-(35) take the following form:

$$q_{xt} - \rho q - 8i\xi a_1\bar{a}_2 + 4q(|a_2|^2 - |a_1|^2) = 0, \tag{36}$$

$$\rho_x + 2(|q|^2)_t + 8(q\bar{a}_1a_2 + \bar{q}a_1\bar{a}_2) = 0, \tag{37}$$

$$a_{1x} + i\xi a_1 - qa_2 = 0, \tag{38}$$

$$a_{2x} - i\xi a_2 + \bar{q}a_1 = 0. \tag{39}$$

**Bilinear forms**

Let us simplify the system (36)-(39). We then obtain

$$q_{xt} - \rho q - 8i\xi a_1\bar{a}_2 + 4q(|a_2|^2 - |a_1|^2) = 0, \tag{40}$$

$$\rho_x + 2(|q|^2)_t + 8(a_{1x} + i\xi a_1)\bar{a}_1 - 8(a_{2x} - i\xi a_2)\bar{a}_2 = 0. \tag{41}$$

With the dependent variable transformations

$$q = \frac{g}{f}, \tag{42}$$

$$\rho = 1 + 2(\ln(f))_{xt}, \tag{43}$$

$$a_1 = \frac{r_1}{f}, \tag{44}$$

$$a_2 = \frac{r_2}{f}, \tag{45}$$

we obtain the bilinear forms for Equation (40)-(41) as

$$\frac{1}{f^2}[D_xD_t(g\cdot f) - gf - 8i\xi r_1\bar{r}_2] - \frac{2gf}{f^4}[D_xD_t(f\cdot f) - 2(|r_2|^2 - |r_1|^2)] = 0,$$

$$\left(\frac{1}{f^2}[D_xD_x(f\cdot f) + 2|g|^2]\right)_t + 8\left([D_x(r_1\cdot f) + i\xi r_1 f]\frac{\bar{r}_1}{f^2}\right) + 8\left([D_x(r_2\cdot f) - i\xi r_2 f]\frac{\bar{r}_2}{f^2}\right) = 0,$$

or

$$D_xD_t(g\cdot f) - gf - 8i\xi r_1\bar{r}_2 = 0, \tag{46}$$

$$D_xD_t(f\cdot f) - 2(|r_2|^2 - |r_1|^2) = 0, \tag{47}$$

$$D_xD_x(f\cdot f) + 2|g|^2 = 0, \tag{48}$$

$$D_x(r_1\cdot f) + i\xi r_1 f = 0, \tag{49}$$

$$D_x(r_2\cdot f) - i\xi r_2 f = 0, \tag{50}$$

where $D_x$ and $D_t$ are the bilinear differential operators defined by

$$D_x^lD_t^n(f\cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x,t)\cdot g(x',t')|_{x'=x,t'=t}$$

with $x'$ and $t'$ as three formal variables, $f(x,t)$ and $g(x',t')$ being two functions, $l$ and $n$ are three

nonnegative integers. To obtain the soliton solutions of equation (46)-(50), we expand $f, g, r_1, r_2$ with respect to a small parameter $\epsilon$ as follows:

$$g(x,y,t) = \epsilon g_1(x,y,t) + \epsilon^3 g_3(x,y,t) + \cdots, \quad (51)$$

$$r_2(x,y,t) = \epsilon r_{21}(x,y,t) + \epsilon^3 r_{23}(x,y,t) + \cdots, \quad (52)$$

$$f(x,y,t) = 1 + \epsilon^2 f_2(x,y,t) + \epsilon^4 f_4(x,y,t) + \cdots, \quad (53)$$

**Results and discussion**

In order to obtain the one-soliton solutions for equations (23)-(26), we choose

$$\begin{aligned} g &= \epsilon g_1, r_2 = \epsilon r_{21}, \\ f &= 1 + \epsilon^2 f_2, \\ r_1 &= 1 + \epsilon^2 r_{12}. \end{aligned} \quad 5)$$

Substituting expression (55) into equations (46)-(50) and collecting the coefficients of the same power of $\epsilon$, we have from equation (46)

$$\epsilon^1: D_x D_t(g_1 \cdot 1) - g_1 + 8i\xi r_{21}^* = 0, \quad (56)$$

$$\epsilon^3: D_x D_t(g_1 \cdot f_2) - g_1 f_2 + 8i\xi r_{12} r_{21}^* = 0, \quad (57)$$

from equation (47)

$$\epsilon^2: D_x D_t(f_2 \cdot 1 + 1 \cdot f_2) - 2(r_{21} r_{21}^* + r_{12} + r_{12}^*) = 0, \quad (58)$$

$$\epsilon^4: D_x D_t(f_2 \cdot f_2) - 2(r_{12} r_{12}^*) = 0, \quad (59)$$

…

from equation (48)

$$\epsilon^2: D_x^2(f_2 \cdot 1 + 1 \cdot f_2) - 2(g_1 g_1^*) = 0, \quad (60)$$

$$\epsilon^4: D_x^2(f_2 \cdot f_2) = 0, \quad (61)$$

…

from equation (49)

$$\epsilon^2: D_x^2(r_{12} \cdot 1 + 1 \cdot f_2) + i\xi(r_{12} + f_2) = 0, \quad (62)$$

$$\epsilon^4: D_x^2(r_{12} \cdot f_2) = 0, \quad (63)$$

…

from equation (50)

$$\epsilon^1: D_x(r_{21} \cdot 1) - i\xi r_{21} = 0, \quad (64)$$

$$\epsilon^3: D_x(r_{21} \cdot f_2) - i\xi r_{21} f_2 = 0, \quad (65)$$

$$r_1(x,y,t) = 1 + \epsilon^2 r_{12}(x,y,t) + \epsilon^4 r_{14}(x,y,t) + \cdots, \quad (54)$$

where $g_j, r_{2j}, r_{1n} (j = 1,3,5,\ldots; n = 2,4,6,\ldots)$ are the complex functions of $x$ and $t$, and $f_n$ is the real function.

Using expressions (51)-(54) and symbolic computation, we obtain the one-soliton solutions for equations (23)-(26).

Solving system of equations (56)-(65), we obtain

$$g_1 = e^{\theta_1},$$

$$r_{21} = A^* e^{\theta_1^*},$$

$$f_2 = B e^{\theta_1 + \theta_1^*},$$

$$r_{12} = S e^{\theta_1 + \theta_1^*},$$

where $A^* = -\frac{1 - k_1^* \omega_1^*}{8i\xi}$, $B = -\frac{1}{(k_1 + k_1^*)^2}$, $S = -\frac{((k_1 + k_1^*) - i\xi)}{((k_1 + k_1^*) + i\xi)(k_1 + k_1^*)^2}$, $\theta_1 = k_1 x + \omega_1 t + \theta_0$, $k_1 = -i\xi, k_1$ is a complex constant.

Setting $\epsilon = 1$ and through expressions (42)-(45), we obtain the one-soliton solutions for the equation (40)-(41) as follows:

$$q = \frac{e^{\theta_1}}{1 + B e^{\theta_1 + \theta_1^*}},$$

$$\rho = 1 + 2(ln(1 + B e^{\theta_1 + \theta_1^*}))_{xt},$$

$$a_1 = \frac{1 + S e^{\theta_1 + \theta_1^*}}{1 + B e^{\theta_1 + \theta_1^*}},$$

$$a_2 = \frac{A^* e^{\theta_1^*}}{1 + B e^{\theta_1 + \theta_1^*}},$$

or with (27) we can obtain

$$q = \frac{e^{\theta_1}}{1 + B e^{\theta_1 + \theta_1^*}},$$

$$\rho = 1 + 2(ln(1 + B e^{\theta_1 + \theta_1^*}))_{xt},$$

$$p = 2\left(\frac{1 + S e^{\theta_1 + \theta_1^*}}{1 + B e^{\theta_1 + \theta_1^*}}\right) \frac{A e^{\theta_1}}{1 + B^* e^{\theta_1 + \theta_1^*}},$$

$$\eta = \left|\frac{1+Se^{\theta_1+\theta_1^*}}{1+Be^{\theta_1+\theta_1^*}}\right|^2 - \left|\frac{A^*e^{\theta_1^*}}{1+Be^{\theta_1+\theta_1^*}}\right|^2.$$

Graphical representations of the one-soliton solutions are presented in Figures 1–3.

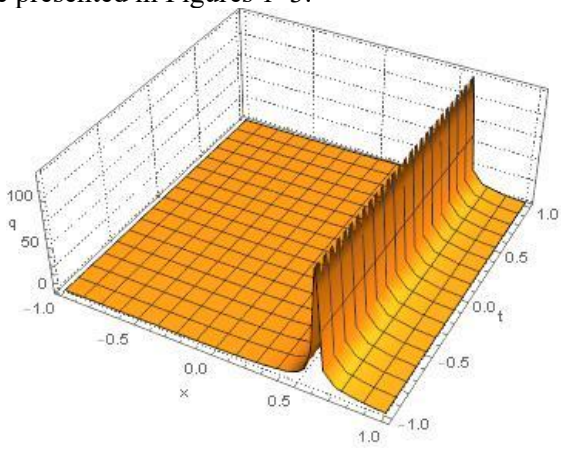



**Figure 1.** One soliton solutions $q$ of the Konno-Oono equation with self-consistent sources

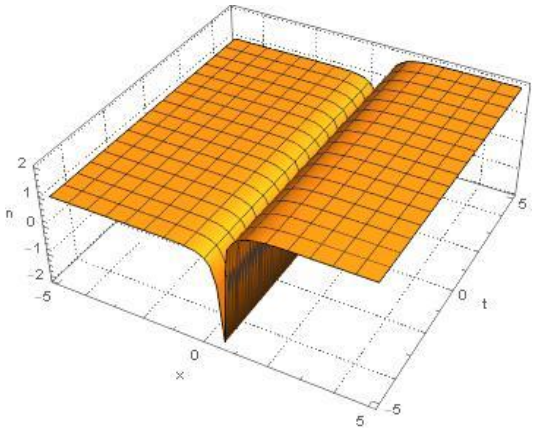



**Figure 2.** - One soliton solutions $\eta$ of the Konno-Oono equation with self-consistent sources

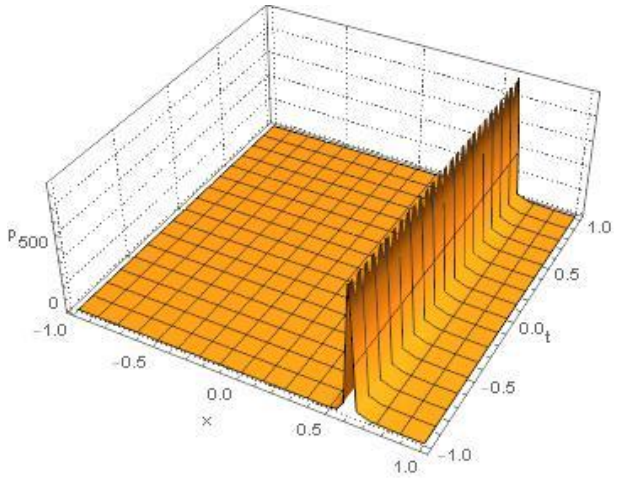



**Figure 3.** - One soliton solutions $p$ of the Konno-Oono equation with self-consistent sources

Figure 1 shows a localized nonlinear wave shaped like a single pulse and maintaining its shape as it propagates. Ultimately, a stable soliton is established (a balance between dispersion and nonlinearity).

Figure 2 interprets the field of soliton *η(x,t)* and describes a localized structure that changes sign. This means that the environment actively reacts to the soliton, and energy is transferred within the system.

Figure 3 shows that p(x,t) characterizes a localized pulsed and phase-sensitive structure. Unlike $q$, it can be complex and contains phase information.

## Conclusion

In this paper, we have investigated the Konno-Oono equation with self-consistent sources (KOESCS), an integrable extension of the coupled dispersionless equation. We derived its Hamiltonian structure by expanding the spectral parameter in inverse powers, yielding an infinite hierarchy of conserved quantities expressed in terms of smooth real functions and their derivatives. This framework underscores the integrable nature of the system, aligning it with broader classes of nonlinear differential equations that exhibit symmetries and conservation laws.

Furthermore, we employed the Hirota bilinear method to construct soliton solutions. By introducing dependent variable transformations and power series expansions, we obtained explicit one-soliton solutions for the fields $q, \rho, p$ and $\eta$. Graphical representations illustrate the localized, stable wave profiles characteristic of solitons in such systems.

Our results extend previous work on the standard Konno-Oono equation and its generalizations, providing new insights into soliton dynamics under self-consistent potentials. These findings may have applications in modeling physical phenomena, such as current-fed strings in magnetic fields or related integrable spin systems. Future research could explore multi-soliton interactions, rogue waves, or connections to other integrable hierarchies like the Heisenberg ferromagnet equations with self-consistent potentials.

**Acknowledgment**

This research is funded by the Science Committee of the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. AP23487348).

**Author Contributions**

**Z.K. Shanina**: Conceptualization, Methodology, Investigation, Writing – Original Draft; **N. Serikbayev**: Formal Analysis, Writing – Review & Editing.

## References

1. K. Yesmakhanova, G. Nugmanova, G. Shaikhova, G. Bekova, R. Myrzakulov, Coupled dispersionless and generalized Heisenberg ferromagnet equations with self-consistent sources: Geometry and equivalence, *Int. J. Geom. Methods Mod. Phys.* **17** (7), 2050104 (2020). https://doi.org/10.1142/S0219887820501042
2. S.-F. Shen, G.-F. Wang, Y.-Y. Jin, and X.-R. Hu, Wadati–Konno–Ichikawa-Type Integrable Systems and Their Constructions, *J. Nonlinear Math. Phys.* **30**, 813 (2023). https://doi.org/10.1007/s44198-022-00102-4
3. L. Bougoffa and S. Bougouffa, Exact Stationary Solution for the Coupled Nonlinear Wadati–Konno–Ichikawa System, *Rom. J. Phys.* **71**, 102 (2026). https://doi.org/10.59277/RomJPhys.2026.71.102
4. A. K. Prykarpatski, Y. A. Prykarpatsky, O. E. Hentosh, and A. Balinsky, The Dispersionless Completely Integrable Heavenly Type Hamiltonian Flows and Their Differential-Geometric Structure, *Ann. Math. Phys.* **2**, 011 (2019). http://dx.doi.org/10.17352/amp.000006
5. E.V. Ferapontov, B. Kruglikov, and V.S. Novikov, Integrability of Dispersionless Hirota-Type Equations and the Symplectic Monge–Ampère Property, *Int. Math. Res. Not.* **2021** (18), 14220 (2021). https://doi.org/10.1093/imrn/rnaa025
6. M. Dunajski and P. Plansangkate, The Quadric Ansatz for the Dispersionless KP Equation and Supersymmetric Einstein–Weyl Spaces, *J. Phys. A: Math. Theor.* **55**, 105202 (2022). https://doi.org/10.1088/1751-8121/ac5160
7. A.K. Prykarpatski, M.N. Hentosh, and Y.A. Prykarpatsky, New Completely Integrable Dispersionless Dynamical System of Heavenly Type Generated by Vector Fields on a Torus, *J. Math. Sci.* **268**, 544 (2022). https://doi.org/10.1007/s10958-022-06077-3
8. E.V. Ferapontov and M. Vermeeren, Lagrangian Multiforms and Dispersionless Integrable Systems, *Lett. Math. Phys.* **115**, 125 (2025). https://doi.org/10.1007/s11005-025-02016-w
9. M. Kamchatnov, Hamiltonian Mechanics of "Magnetic" Solitons in Two-Component Bose–Einstein Condensates, *Studies in Applied Mathematics* **153** (4) e12757 (2024). https://doi.org/10.1111/sapm.12757
10. A. Stepanenko, Spectral inclusion and pollution for a class of dissipative perturbations, *J. Math. Phys.* **62**, 013501 (2021). https://doi.org/10.1063/5.0028440
11. G.T. Bekova, G.N. Shaikhova, K.R. Yesmakhanova, R. Myrzakulov, Conservation laws for two dimensional nonlinear Schrödinger equation, *AIP Conference Proceedings* **2159**, 030003 (2019). https://doi.org/10.1063/1.5127468
12. E.D. Silva; J.C. de Albuquerque; M.L. Silva, Periodic and asymptotically periodic quasilinear elliptic systems, *J. Math. Phys.* **61**, 091501 (2020). https://doi.org/10.1063/5.0012134
13. R. Lin, X. Liu and Y. Zeng, The KP hierarchy with self–consistent sources: construction, Wronskian solutions and bilinear identities, *J. Phys.: Conf. Ser.* **538**, 012014 (2014). https://doi.org/10.1088/1742-6596/538/1/012014
14. G. He, Z. Yang, T. Xu, M. Wang, Bi-Hamiltonian structure and conservation laws of the coupled Boussinesq hierarchy, *Appl. Math. Lett.* **171**, 109652 (2025). https://doi.org/10.1016/j.aml.2025.109652
15. Wen-Xiu Ma, Four-component integrable hierarchies and their Hamiltonian structures, *Communications in Nonlinear Science and Numerical Simulation,* **126**, 107460 (2023). https://doi.org/10.1016/j.cnsns.2023.107460
16. M.A. E. Abdelrahman, H. A. Alkhidhr, Fundamental Solutions for the New Coupled Konno–Oono Equation in Magnetic Field, *Results Phys.* **19**, 103445 (2020). DOI: 10.1016/j.rinp.2020.103445
17. E.H.M. Zahran and A. Bekir, New Diverse Soliton Solutions for the Coupled Konno–Oono Equations, *Opt. Quantum Electron.* **55**, 112 (2023). https://doi.org/10.1007/s11082-022-04376-2
18. Y. Chahlaoui, A. Ali, J. Ahmad, and S. Javed, Dynamical Behavior of Chaos, Bifurcation Analysis and Soliton Solutions to a Konno–Onno Model, *PLoS One* **18**, e0291197 (2023). https://doi.org/10.1371/journal.pone.0291197
19. T. S. Shaikh, M. Z. Baber, N. Ahmed, On the Soliton Solutions for the Stochastic Konno–Oono System in Magnetic Field with the Presence of Noise, *Mathematics* **11**, 1472 (2023). https://doi.org/10.3390/math11061472
20. M. Iqbal, D. Yaro, W.O. Apeanti, Optical Soliton Solutions of the Nonlinear Coupled Konno–Oono System by Using the New Analytical Approach and Modulation Instability Analysis, *Sci. Rep.* **16**, 20957 (2026). https://doi.org/10.1038/s41598-026-51821-3
21. E.A. Az-Zo'bi, E. Hussain, M. Iqbal, Chaotic, Bifurcation, Sensitivity, Modulation Stability Analysis, and Optical Soliton Structure to the Nonlinear Coupled Konno–Oono System in Magnetic Field, *AIP Adv.* **15**, 095103 (2025). https://doi.org/10.1063/5.0291009
22. U. Asghar, M.I. Asjad, Y. Alsayaad, and Y. S. Hamed, Algebraic Approaches for Deriving Soliton Solutions and Analyzing the Stability of Coupled Konno–Oono System Arising in Magnetic Field, *J. Theor. Appl. Phys.* **19** (1) (2025). https://doi.org/10.57647/j.jtap.2025.1901.15

23. F.A. Abdullah, Md. T. Islam, J. F. Gómez-Aguilar & Md. Ali Akbar, Impressive and Innovative Soliton Shapes for Nonlinear Konno–Oono Equations, *Opt. Quantum Electron*. **55**, 69 (2023). https://doi.org/10.1007/s11082-022-04308-0

24. R. Kumar, A. Kumar, Some invariant solutions of coupled Konno-Oono equations arising in electromagnetic and quantum fields, *Physica Scripta.* **97** (7), 075501 (2022). https://doi.org/10.1088/1402-4896/ac71e1

25. Z. Shanina, Dispersionless Limit of the Konno–Oono Equation, *Bull. L. N. Gumilyov Eurasian Natl. Univ. Phys. Astron. Ser.* **129**, 93 (2019). https://doi.org/10.32523/2616-6836-2019-129-4-93-98

26. Z. Shanina, R. Myrzakulov, Soliton surfaces induced by the coupled integrable dispersionless equation with self-consistent sources, *Journal of Physics: Conference Series.* **1391**, 012104 (2019). https://doi.org/10.1088/1742-6596/1391/1/012104

27. G.T. Bekova, G.N. Shaikhova, K.R. Yesmakhanova, R. Myrzakulov, Darboux transformation and soliton solution for generalized Konno-Oono equation, *Journal of Physics: Conference Series* **1416**, 012003 (2019). https://doi.org/10.1088/1742-6596/1416/1/012003

***Information about authors:***

*Zamzagul Shanina (corresponding author) – master's in physics, Senior Lecturer of the Department of Physics, Faculty of Physics and Mathematics, K. Zhubanov Aktobe Regional University, (Aktobe, Kazakhstan; e-mail: zshanina@zhubanov.edu.kz).*

*Nurzhan Serikbayev – PhD, Associate Professor of the Department of General and Theoretical Physics, Institute of Physical and Technical Sciences, L. N. Gumilyov Eurasian National University, (Astana, Kazakhstan; e-mail: serikbayev_ns@enu.kz).*

***Авторлар туралы мәлімет:***

*Замзагул Куатовна Шанина (автор-корреспондент) - физика магистрі, физика кафедрасының аға оқытушысы, физика-математика факультеті, Қ. Жұбанов атындағы Ақтөбе өңірлік университеті, (Ақтөбе, Қазақстан, e-mail: zshanina@zhubanov.edu.kz)*

*Нуржан Серикбаев - PhD, жалпы және теориялық физика кафедрасының қауымдастырылған профессоры, Физика - техникалық ғылымдар институты, Л.Н. Гумилев атындағы Еуразия ұлттық университеті, (Астана, Қазақстан, e-mail: serikbayev_ns@enu.kz)*

***Сведения об авторах:***

*Замзагул Куатовна Шанина (автор-корреспондент) - магистр физики, старший преподаватель кафедры физики, физико-математический факультет, Актюбинский региональный университет имени К.Жубанова, (Актобе, Казахстан, e-mail: zshanina@zhubanov.edu.kz).*

*Нуржан Серикбаев - PhD, ассоциированный профессор кафедры общей и теоретической физики, институт физико – технических наук, Евразийский национальный университет имени Л. Н. Гумилёва, (Астана, Казахстан, e-mail: serikbayev_ns@enu.kz).*

***Article history:*** *received: 22 June 2026; accepted: 03 September 2026.*

***Мақала тарихы****: түсті: 22 мауысым 2026; қабылданды: 03 қыркүйек 2026.*

***История статьи:*** *поступила: 22 июня 2026; принята: 03 сентября 2026.*